Free crossed resolutions of groups and presentations of modules of identities among relations

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Abstract

We give formulae for a module presentation of the module of identities among relations for a presentation of a group, in terms of information on 0- and 1-combings of the Cayley graph. This is seen as a special case of extending a partial free crossed resolution of a group given a partial contracting homotopy of its universal cover.

KEYWORDS: identities among relations, crossed modules, crossed complexes, resolutions of groups.
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Introduction

The initial motivation for this work was to determine algebraically a presentation for the G-module $\pi(\mathcal{P})$ of identities among relations for a presentation $\mathcal{P} = \langle X|\omega : R \to F(X)\rangle$ of a group G. Here we regard R as a set disjoint from $F(X)$ and ω gives the corresponding element of $F(X)$. Recall that $\pi(\mathcal{P})$ is given algebraically as the kernel of $\delta_2 : C(R) \to F(X)$, the free crossed module of the presentation, and is given geometrically as $\pi_2(K(\mathcal{P}))$, the second homotopy group of the cell complex of the presentation.

Our main results imply a formula as follows:

Theorem A *The module $\pi(\mathcal{P})$ is generated as G-module by elements*

$$\delta_3[g,r] = (k_1(g,\omega r))^{-1}\, r^{(\sigma g)^{-1}}$$

for all $g \in G, r \in R$, where (i) $\sigma : G \to F(X)$ is a section of the quotient mapping $\varphi : F(X) \to G$, (ii) k_1 is a morphism $F(\widetilde{X}) \to C(R)$ from the free groupoid on $\widetilde{X}$, the Cayley graph of the presentation, to the free crossed module of the presentation, such that $\delta_2 k_1(g,x) = (\sigma g)x(\sigma(g(\varphi x)))^{-1}$, for all $x \in X, g \in G$.

The identities $\delta_3[g,r]$ may be seen as separation elements in the geometry of the *Cayley graph with relators*, as defined in sections 1,3. The main feature of the theorem is that these elements generate all identities, since it is easy to see from properties (i), (ii) and the first crossed module rule that these elements are all identities.

The identities $\delta_3[g,r]$ will be seen to arise from a boundary mapping $\delta_3 : C_3(I) \to C(R)$ from the free G-module on a set I bijective with $G \times R$, with basis elements written $[g,r], g \in G, r \in R$. The set $\delta_3(I)$ is usually not a minimal set of generators (many of them may even be trivial). So we suppose given a subset J of I, determining a free G-module $C_3(J)$, and minimal with respect to the property that $\delta_3(J)$ also generates $\pi(\mathcal{P})$, and then seek relations among these generators $\delta_3(J)$.

Theorem B *A G-module generating set of relations among these generators $\delta_3(J)$ of $\pi(\mathcal{P})$ is given by*

$$\delta_4[g,\gamma] = -k_2(g,\delta_3\gamma) + \gamma.g^{-1}$$

for all $g \in G, \gamma \in J$, where $k_2 : C(\widetilde{R}) \to C_3(J)$ is a morphism from the free crossed $F(\widetilde{X})$-module on $\widetilde{\delta}_2 : G \times R \to F(\widetilde{X})$ such that k_2 kills the operation of $F(\widetilde{X})$ and is determined by a choice of writing the generators $\delta_3[g,r] \in \delta_3(I)$ for $\pi(\mathcal{P})$ in terms of the elements of $\delta_3(J)$.

It will be noted that both these results use the language of groupoids which is convenient for encoding the graphical information. We use in an essential way morphisms from a groupoid to a group.

In section 1 we shall explain the terms in these theorems in sufficient detail for the reader to follow an explicit calculation for the standard presentation of the group S_3 in section 2. We give this example because it is sufficiently complex to illustrate important features of the calculations, and sufficiently simple that the calculations can be carried out by hand.

In this example, Theorem A gives 18 generators for the module $\pi(\mathcal{P})$; we show this number can be reduced to 4. This minimal set of generators was already known. The rewriting involved in this reduction process is then used to construct the next level of syzygies, using Theorem B. This yields initially 24 relations among identities which are then shown to reduce to 5 independent ones. We are

not aware of any previous determination of the relations among these identities. These calculations have been extended by hand, but with different choices, by two further stages in [23].

The reader will notice the analogy between the formulae in these theorems – they are in fact special cases of Corollary 9.3, which computes higher order syzygies inductively. The context of that result is that of free crossed resolutions, universal covers of crossed resolutions, and contracting homotopies of such universal covers. Once this machinery is set up, the result becomes almost tautologous. It states that the pair consisting of a partial free crossed resolution and a partial contracting homotopy of its universal cover can be extended by one step, and hence indefinitely.

A sequel to this paper by Heyworth and Wensley [25] will show how the part of the procedure required for Theorem A can be implemented as a 'logged Knuth-Bendix procedure'. A further paper by Heyworth and Reinert [24] will show how generalised Gröbner basis procedures for integral group rings can implement the reduction process required for Theorem B, and so allow a wide range of computations.

The partial contracting homotopies are given by functions h_i for $i < n$ with appropriate properties. In fact h_0 corresponds to a 0-combing, and h_1 is analogous to a 1-combing; from these we obtain the functions σ, k_1 of the theorems. The algebra of such functions is shown to be nicely handled in the context of the free groupoid $F(\widetilde{X})$ on the Cayley graph and the free crossed module $\widetilde{\delta}_2 : C(\widetilde{R}) \to F(\widetilde{X})$. We show that this crossed module is the fundamental crossed module of the universal cover of the geometric 2-complex of the presentation. The groupoid approach is required to utilise all the vertices of the universal cover.

A computation of the module of identities among relations for the presentation $\langle X|R\rangle$ of the group G could be seen in the context of chain complexes and resolutions of modules as that of computing an extension of the partial resolution of $\mathbb{Z}$

$$(\mathbb{Z}G)^R \to (\mathbb{Z}G)^X \to \mathbb{Z}G \to \mathbb{Z}$$

where the first morphism is given by the Whitehead-Fox derivative $(\partial r/\partial x)$ [37, Lemma 8], [20]. The process of extending a partial resolution is more difficult than that of just giving a resolution. There is in fact considerable work on constructing resolutions of groups, some of it for 2-groups mod 2, and other results using homological perturbation theory, particularly by Larry Lambe and colleagues [32]. It is not clear how these methods apply to the problem of extending partial resolutions. Work of Groves [21] constructs a resolution from a complete rewrite system for a monoid presentation of the group, rather than directly from a group presentation. However, as mentioned above, complete rewrite systems are relevant to the computation of k_1.

It is interesting to compare our methods with the methods of pictures for calculating the generators of $\pi(\mathcal{P})$ (see for example [16, 28, 33]). These methods use nicely the geometry of the relations, they have been very successful in this field, and can be more efficient than ours for this dimension. However they seem more difficult to carry out in higher dimensions, for the following reasons.

The picture methods use 2-dimensional rewrite rules to reduce spherical elements to a combination of standard elements. The full information on the way these rewrites are used in a particular example is essentially 3-dimensional, and it can thus be difficult to visualise or to record combinations of such rewrites, and their dependencies. For our purposes this rewriting information must be recorded completely (see Tables 2,4 of section 2) since it is used to construct the next stage of a contracting homotopy; this use of the complete record is one reason for the apparently cumbersome nature of

the calculations. Thus there are problems in extending the picture method to determine 3-syzygies, whereas our purely algebraic method is essentially uniform over dimensions, giving rise mainly to computational problems. This suggests that in dimension 2 our methods should be seen as complementary to those of pictures.

The method of pictures has also been applied successfully to determine generators for the module of identities among relations for various constructions on groups. By contrast the only general construction on crossed resolutions which has so far been applied is the tensor product [13, 17, 35] – given free crossed resolutions C, C' of two groups G, H, the tensor product $C \otimes C'$ gives a free crossed resolution of their product $G \times H$, and so a presentation of the module of identities for the standard combined presentation of the product. An application is in [17].

There are three basic planks in our approach.

(i) *Crossed complexes*

Crossed complexes form an analogue of chain complexes but with non abelian features in dimensions 1 and 2. These features allow crossed complexes to combine many of the advantages of chain complexes with an ability to contain the information involved in a presentation of a group. So one can model many of the standard techniques of homological algebra, such as uniqueness up to homotopy of a free crossed resolution. Further, this technique may be combined with a non abelian version of the traditional notion of 'chains of syzygies'; this version takes account of the facts that free groups are non abelian, and that a normal subgroup N of a group F is in general non abelian, and admits an operation of F on N which is crucial in discussing presentations. Crossed complexes, unlike chain complexes, allow for 'free' models of this inclusion $N \to F$ (see [16]), and so give an intuitive algebraic model of chains of syzygies in this non abelian case. An account of uses of crossed complexes up to 1981 is given in [12].

A small free crossed resolution is convenient for calculations of non abelian extensions [17] and of the cohomology class of a crossed module [18, 19]. A free crossed resolution C of G determines a free $\mathbb{Z}G$ resolution ΔC of $\mathbb{Z}$ in the usual sense [37, 15]. The crossed resolution C with its free basis carries more information than ΔC, for example it includes a presentation of G.

(ii) *Algebraic models of the geometry of covering spaces*

Philip Higgins pointed out in 1964 [26] how presentations of groupoids could be applied to group theory. The geometric basis of the argument is that the theory of covering spaces is more conveniently handled if one uses groupoids rather than groups, since there is a purely algebraic notion of *covering morphism of a groupoid* which nicely models the geometry (see [4]). Covering morphisms of a group or groupoid G are equivalent to operations of G on sets.

In the same way, to apply crossed complexes to covering spaces we require crossed complexes of groupoids not just of groups. Such general crossed complexes were also found essential in [11] for certain higher order Van Kampen Theorems, so the basic definitions and applications are already known. This allows us to bring in techniques not only of presentations of groupoids, as discussed in [26], but also of free crossed resolutions determined by such a presentation.

In effect, we are giving a suitable *algebraic* framework in which to place the geometry of the Cayley graph of a generating set of the group, but including the relations as well as the generators of the presentation, and indeed including higher order syzygies, as these are constructed. This algebraic framework also models conveniently the geometry of the universal cover of a cell complex.

A crucial tool for our methods is the fact that a covering crossed complex of a free crossed complex is again a free crossed complex, on the 'covering generators' (Theorem 8.2). This models the geometry of CW-complexes. The result is crucial because it enables us to define morphisms and homotopies by their values on the free generators. Our proof relies on a result of Howie [29].

(iii) *Contracting homotopies*

The key point is that the previous techniques allow us to discuss free crossed resolutions of contractible groupoids, for example the universal covering groupoid of the original group. A crossed resolution of a contractible groupoid will have contracting homotopies, and our method proceeds by the construction of such homotopies. This method is applied to truncated crossed complexes and in particular to crossed modules. The usual slogan *choose generators for the kernel* and so kill homotopy groups, fails to tell us how to choose these generators. Instead we construct a crossed complex whose universal cover is a *home for a contracting homotopy.* This 'tautologously' yields generators of kernels.

In order to make this method clear, we need the basics of the theory of presentations and of identities among relations for groupoids. We give the key features, largely without proofs, in section 3.

The basic theory of crossed complexes and their covering morphisms that we need is presented in sections 6–8. Finally, the notion of homotopy for crossed complexes is presented in section 9.

Our method yields a resolution dependent functorially on the presentation. However a count of the numbers of generators in various dimensions shows that the module resolution obtained from our crossed resolution by the process of [14] is not the same as the Gruenberg resolution [22]. We are grateful to Justin Smith for pointing out this reference.

More generally, we can obtain a free crossed resolution dependent functorially on the first n stages of a free crossed resolution, with basis up to this stage.

In the final section we show how these methods give rise to the standard crossed resolution of a group G, and to a small crossed resolution of a finite cyclic group. In each case, the information on the contracting homotopy determines the resolution.

We would like to thank Anne Heyworth and Emma Moore for discussions on this material which led to the exposition in sections 1, 10.

1 The computational procedure

The purpose of this section is to state the computational procedure in as direct a way as we can. The theoretical underpinning is left to later sections. We hope this will make it easier for the reader.

Let $\mathcal{P} = \langle X | \omega : R \to F(X) \rangle$ be a presentation of a group G. The advantages of using the function ω are (i) to allow for the possibility of repeated relations, and (ii) to distinguish between an element $r \in R$ and the corresponding element $w(r) \in F(X)$. We shall be concerned with the following diagram, in which $p_0 \widetilde{\varphi} = \varphi p_1, p_1 \widetilde{\delta}_2 = \delta_2 p_2, p_2 \widetilde{\delta}_3 = \delta_3 p_3$. The parts of this diagram will be developed below:

$$
\begin{tikzcd}
G & G & G & G \arrow[dl, bend right, "h_0"'] \\
C_3(\widetilde{I}) \arrow[u, "\beta"] \arrow[r, shift left, "\widetilde{\delta}_3"] \arrow[d, "p_3"'] & C(\widetilde{R}) \arrow[u, "\beta"] \arrow[l, shift left, "h_2"] \arrow[r, shift left, "\widetilde{\delta}_2"] \arrow[d, "p_2"'] & F(\widetilde{X}) \arrow[u, "\beta"] \arrow[l, shift left, "h_1"] \arrow[r, "\widetilde{\varphi}"] \arrow[d, "p_1"'] & \widetilde{G} \arrow[u, shift right, "\delta^0"] \arrow[u, shift left, "\delta^1"'] \arrow[d, "p_0"'] \\
C_3(I) \arrow[r, "\delta_3"'] & C(R) \arrow[r, "\delta_2"'] & F(X) \arrow[r, "\varphi"'] & G
\end{tikzcd}
\tag{1}
$$

1.1 $\varphi : F(X) \to G$ is the canonical morphism from the free group on X to G given by the set of generators.

1.2 $\delta_2 : C(R) \to F(X)$ is the free crossed $F(X)$-module on the function ω.

Thus the elements of $C(R)$ are 'formal consequences'

$$
c = \prod_{i=1}^{n} (r_i^{\varepsilon_i})^{u_i}
$$

where $n \geqslant 0, r_i \in R, \epsilon_i = \pm 1, u_i \in F(X)$, $\delta_2(r^\epsilon)^u = u^{-1}(\omega r)^\epsilon u$, subject to the crossed module rule $ab = ba^{\delta_2 b}, a, b \in C(R)$. For information on crossed modules, and particularly free crossed modules, see for example [16, 28, 7].

Let $N = \text{Ker } \varphi$. Then $\delta_2(C(R)) = N$. Of course it is the kernel $\pi(\mathcal{P})$ of δ_2, the G-module of identities among relations, that we wish to calculate. For this we require algebraic analogues of methods of covering spaces, and so use the language of groupoids. Our convention is that the product of elements (arrows) $a : g \to g', a' : g' \to g''$ in a groupoid Γ is written $aa' : g \to g''$, and $\Gamma(a)$ denotes the object group of Γ at a, i.e. the set of arrows $a \to a$ with the induced group structure.

1.3 $p_0 : \widetilde{G} \to G$ is the universal covering groupoid of the group G. The objects of $\widetilde{G}$ are the elements of G, and an arrow of $\widetilde{G}$ is a pair $(g, g') \in G \times G$ with source $g = \delta^0(g, g')$ and target $gg' = \delta^1(g, g')$. The projection morphism p_0 is given by $(g, g') \mapsto g'$.

1.4 $\widetilde{X}$ is the *Cayley graph* of the pair (G, X). Its objects are the elements of G and its arrows are pairs $(g, x) \in G \times X$ with source $g = \delta^0(g, x)$ and target $g(\varphi x) = \delta^1(g, x)$, also written $\beta(g, x)$.

1.5 $F(\widetilde{X})$ is the *free groupoid* on $\widetilde{X}$. Its objects are the elements of G and its arrows are pairs $(g, u) \in G \times F(X)$ with source g and target $g(\varphi u)$. We also write $\beta(g, u) = g(\varphi u)$. The multiplication is given by $(g, u)(g(\varphi u), v) = (g, uv)$. The morphism $\widetilde{\varphi}$ is given by $(g, u) \mapsto (g, \varphi u)$. The morphism p_1 is given by $(g, u) \mapsto u$. It maps the object group $F(\widetilde{X})(1)$ isomorphically to N.

As we shall see in section 6, $\widetilde{G} \to G$ is the covering morphism corresponding to the trivial subgroup of G, and $F(\widetilde{X}) \to F(X)$ is the covering morphism corresponding to the subgroup N of $F(X)$.

1.6 $\widetilde{R} = G \times R$ and $\widetilde{\delta}_2 : C(\widetilde{R}) \to F(\widetilde{X})$ is the *free crossed* $F(\widetilde{X})$*-module* on $\widetilde{\omega} : \widetilde{R} \to F(\widetilde{X})$, $(g,r) \mapsto (g, \omega(r))$. Then $C(\widetilde{R})$ is the disjoint union of groups $C(\widetilde{R})(g), g \in G$, all mapped by p_2 isomorphically to $C(R)$. Elements of $C(\widetilde{R})(g)$ are pairs $(g,c) \in \{g\} \times C(R)$, with multiplication $(g,c)(g,c') = (g,cc')$. The (partial) action of $F(\widetilde{X})$ is given by $(g,c)^{(g,u)} = (g(\varphi u), c^u)$. The boundary $\widetilde{\delta}_2$ is given by $(g,c) \mapsto (g, \delta_2 c)$. The morphism $p_2 : C(\widetilde{R}) \to C(R)$ is given by $(g,c) \mapsto c$.

If $(g,c) \in C(\widetilde{R})(g)$ we write $\beta(g,c) = g$; we call β the *base point map*. The elements of $C(\widetilde{R})(g)$ are also all 'formal consequences'

$$(g,c) = \prod_{i=1}^{n} ((g_i, r_i)^{\varepsilon_i})^{(g_i,u_i)} = \prod_{i=1}^{n} (g, (r_i^{\varepsilon_i})^{u_i}) = (g, \prod_{i=1}^{n} (r_i^{\varepsilon_i})^{u_i})$$

where $n \geqslant 0$, $r_i \in R$, $\epsilon_i = \pm 1$, $u_i \in F(X)$, $g_i \in G$, $g_i(\varphi u_i) = g$, subject to the crossed module rule $ab = ba^{\widetilde{\delta}_2 b}, a, b \in C(\widetilde{R})$. Here the first form of the product is useful geometrically, and the last is useful computationally.

In effect, we are giving first a presentation $\langle \widetilde{X} | \widetilde{\omega} : \widetilde{R} \to F(\widetilde{X}) \rangle$ of the groupoid $\widetilde{G}$ [26], and second the free crossed module corresponding to this presentation.

The proof that the construction given in 1.6 does give the free crossed module as claimed is given in theorem 8.2.

We now construct $C_3(I)$ and its cover $C_3(\widetilde{I})$.

1.7 Let I be a set in one-to-one correspondence with $G \times R$ with elements written $[g,r], g \in G, r \in R$. Let $C_3(I)$ be the free G-module on I.

1.8 Let $C_3(\widetilde{I})$ be the free $\widetilde{G}$-module on $\widetilde{I} = G \times I$. Then $C_3(\widetilde{I})$ is the disjoint union of abelian groups $C(\widetilde{I})(g), g \in G$, all mapped by p_3 isomorphically to $C_3(I)$. Elements of $C_3(\widetilde{I})(g)$ are pairs $(g,i) \in \{g\} \times C_3(I)$ with addition $(g,i) + (g,i') = (g, i+i')$. The (partial) action of $\widetilde{G}$ on $C_3(\widetilde{I})$ is given by $(g,i).(g,g') = (gg', i.g')$.

The construction of δ_3 (and hence of $\widetilde{\delta}_3$) requires some choices.

1.9 Choose a section $\sigma : G \to F(X)$ of φ such that $\sigma(1) = 1$, and write $\bar{\sigma}(g) = \sigma(g)^{-1}$. Then σ determines a function $h_0 : G \to F(\widetilde{X})$ by $g \mapsto (g, \bar{\sigma} g)$. Thus $h_0(g)$ is a path $g \to 1$ in the Cayley graph $\widetilde{X}$.

Remark 1.10 The choice of h_0 is often, but not always, made by choosing a maximal tree in the graph $\widetilde{X}$ – such a choice is equivalent to a choice of Schreier transversal for the subgroup $N = \mathrm{Ker}\ \varphi$ of $F(X)$. A different choice of h_0 is used in subsection 10.1 for the standard crossed resolution.

For each arrow (g,x) of $\widetilde{X}$ the element $\rho(g,x) = (h_0 g)^{-1}(g,x)h_0(g(\varphi x))$ is a loop at 1 in $F(\widetilde{X})$ and so is in the image of $\widetilde{\delta}_2$.

1.11 For each arrow (g,x) of $\widetilde{X}$ choose an element $h_1(g,x) \in C(\widetilde{R})(1)$ such that

$$\widetilde{\delta}_2(h_1(g,x)) = \rho(g,x).$$

Then h_1 extends uniquely to a morphism $h_1 : F(\widetilde{X}) \to C(\widetilde{R})(1)$ such that for all arrows (g,u) of $F(\widetilde{X})$

$$\widetilde{\delta}_2(h_1(g,u)) = (h_0 g)^{-1}(g,u)h_0(g(\varphi u)). \tag{2}$$

It follows that $h_1(h_0(g)) = 1, g \in G$.

Remark 1.12 The choice of h_1 is equivalent to choosing a representation as a consequence of the relators R for each element of N, given as a word in the elements of X. There is no algorithm for such a choice. It will be shown in [25] how a 'logged Knuth-Bendix procedure' will give such a choice when the monoid rewrite system determined by R may be completed, and that this allows for an implementation of the determination of h_1.

The morphism k_1 of Theorem A of the Introduction is simply the composition $p_2 h_1$.

1.13 Define $\delta_3 : C_3(I) \to C(R)$ by

$$\delta_3[g,r] = p_2\left((h_1(g,w(r)))^{-1}\right)\ r^{\bar{\sigma}g}. \tag{3}$$

It follows from equation (2) that $\delta_2\delta_3 = 0$, and so the given values $\delta_3[g,r]$ lie in the G-module $\pi(\mathcal{P})$. This implies that δ_3 is well defined on $C_3(I)$ by its values on the set I of free module generators.

1.14 Let $\widetilde{\delta}_3 : C_3(\widetilde{I}) \to C(\widetilde{R})$ be the $\widetilde{G}$-morphism given by $\widetilde{\delta}_3(g,d) = (g,\delta_3 d)$. Let $h_2 : C(\widetilde{R}) \to C_3(\widetilde{I})(1)$ be the groupoid morphism killing the operation of $F(\widetilde{X})$ (i.e. $h_2((g,c)^{(g,u)}) = h_2(g,c)$ for all $(g,c) \in C(\widetilde{R}), u \in F(X)$) and satisfying $(g,r) \mapsto (1,[g,r]), (g,r) \in G \times R$. Then for all $g \in G, c \in C(R)$

$$\widetilde{\delta}_3 h_2(g,c) = (h_1\widetilde{\delta}_2(g,c))^{-1}\ (1, c^{\bar{\sigma}g}). \tag{4}$$

1.15 Proof of Theorem A Equations (2), (4) show that $\widetilde{\delta}_2\widetilde{\delta}_3 = 0$, and so the elements $p_2(\widetilde{\delta}_3 h_2(g,c))$ do give identities. On the other hand, if $c \in C(R)$ and $\delta_2 c = 1$, then by equation (4), $(1,c) = \widetilde{\delta}_3 h_2(1,c)$, and so $c = \delta_3(d)$ for some d. Theorem A of the Introduction is an immediate consequence, with $k_1 = p_2 h_1$. □

However some of the elements of $\delta_3(I)$ may be trivial, and others may depend $\mathbb{Z}G$-linearly on a smaller subset. That is, there may be a proper subset J of I such that $\delta_3(J)$ also generates the module $\pi(\mathcal{P})$. Then for each element $i \in I \setminus J$ there is a formula expressing $\delta_3 i$ as a $\mathbb{Z}G$-linear combination of the elements of $\delta_3(J)$. These formulae determine a $\mathbb{Z}G$-retraction $\mathsf{r} : C_3(I) \to C_3(J)$ such that for all $d \in C_3(I), \delta_3(\mathsf{r}d) = \delta_3(d)$. So we replace I in the above diagram by J, replacing the

boundaries by their restrictions. Further, and this is the crucial step, we replace h_2 by $h'_2 = \mathsf{r}' h_2$ where $\mathsf{r}' : C_3(\widetilde{I})(1) \to C_3(\widetilde{J})(1)$ is mapped by p_3 to r.

This $h'_2 : C(\widetilde{R}) \to C_3(\widetilde{J})(1)$ is now used to continue the above construction, by defining $C_4(\bar{J})$ to be the free G-module on elements written $[g, d] \in \bar{J} = G \times J$, with

$$\delta_4[g, d] = -p_3(h'_2(g, \delta_3 d)) + d.g^{-1}. \tag{5}$$

These boundary elements give generators for the relations among the generators $\delta_3(J)$ of $\pi(\mathcal{P})$.

1.16 Proof of Theorem B This is a similar argument to the proof of Theorem A, using equation (5), and setting $k_2 = p_3 h'_2$. □

Remark 1.17 In the above we have defined morphisms and homotopies by their values on certain generators, and so it is important for this that the structures be free. For example, h'_2 is defined by its values on the elements $(g, r) \in G \times R$. So, noting that h_2 kills the operation of $F(\widetilde{X})$, we calculate for example $h'_2(g, r^u s^v) = h'_2(g(\varphi u)^{-1}, r) + h'_2(g(\varphi v)^{-1}, s)$. In this way the formulae reflect the choices made at different parts of the Cayley graph in order to obtain a contraction.

The freeness of $F(\widetilde{X})$ was proved by Higgins in [26]. Our proof for $C(\widetilde{R})$ uses a result of Howie, as we shall see later.

Remark 1.18 The determination of minimal subsets J of I such that $\delta_3 J$ also generates $\pi(\mathcal{P})$ is again not straightforward. Some dependencies are easy to find, and others are not. A basic result due to Whitehead [37] is that the abelianisation map $C(R) \to (\mathbb{Z}G)^R$ maps $\pi(\mathcal{P})$ isomorphically to the kernel of the Whitehead-Fox derivative $(\partial r/\partial x) : (\mathbb{Z}G)^R \to (\mathbb{Z}G)^X$. Hence we can test for dependency among identities by passing to the free $\mathbb{Z}G$-module $(\mathbb{Z}G)^R$, and we use this in the next section. For bigger examples, this testing can be a formidable task by hand. An implementation of Gröbner basis procedures for finding minimal subsets which still generate is described in [24].

2 Syzygies of levels 2 and 3 for the standard presentation of $\mathbf{S_3}$

We illustrate the above method in this section with the standard presentation of the six element group S_3. This is chosen as perhaps the first interesting example which can still be done by hand, and because it does illustrate all the above points. While our set of generators of the module of identities for this presentation is known, we are not aware of previous calculations of the relations between these generators.

The group presentation $\langle x, y \mid x^3, y^2, xyxy\rangle$ determines the symmetric group S_3 on three symbols. Let $X = \{x, y\}$ and let $F = F(X)$ be the free group on X. Let $R = \{r, s, t\}$ and let $\omega : R \to F$ be given by

$$\omega r = x^3, \omega s = y^2, \omega t = xyxy.$$

Let $\varphi : F \to S_3$ be the epimorphism determined by x, y, and let $N = \mathrm{Ker}\ \varphi$. So we have the free crossed module $\delta_2 : C(R) \to F$.

Now we set up the corresponding diagram (1) of the previous section. We think of each element $(g,r) \in \widetilde{R}$ as filling a 2-cell in the Cayley graph $\widetilde{X}$. Thus in this example, each relator, i.e. each element of R, is covered six times in the universal cover. We also see in this situation the rôle of relations which are proper powers. The covers of the element r of R separate into two classes, namely

$$\{(1,r),(\varphi x,r),(\varphi x^2,r)\}, \qquad \{(\varphi y,r),(\varphi yx,r),(\varphi yx^2,r)\}.$$

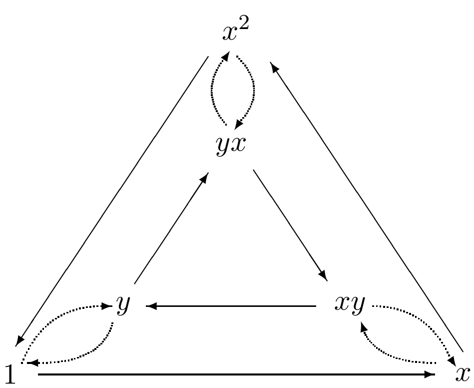
An element of one of these classes has boundary the same 'triangle' in $F(\widetilde{X})$ as the other elements, but with a different starting point. Similarly, the relation ωt is of order 2 and so the covers of t divide into three classes each with 2 elements. A similar statement holds for s.

We now have to choose $\sigma : S_3 \to F(X)$. For this, choose a maximal tree T in the directed graph $\widetilde{X}$. The choice of T is well known to be equivalent to the choice of a Schreier transversal for N in F. For this example, we choose the tree T to be given by the elements

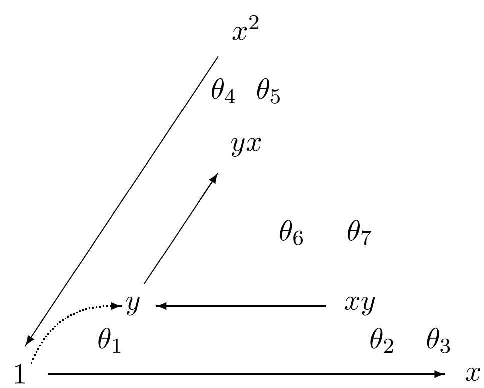
$$(1,y),\ (1,x),\ (\varphi x^2,x),\ (\varphi y,x),\ (\varphi(xy),x).$$

The remaining elements of $\widetilde{X}$ we label as

$$\theta_1 = (\varphi y,y),\ \theta_2 = (\varphi x,y),\ \theta_3 = (\varphi xy,y),\ \theta_4 = (\varphi yx,y), \tag{6}$$
$$\theta_5 = (\varphi x^2,y),\ \theta_6 = (\varphi yx,x),\ \theta_7 = (\varphi x,x). \tag{7}$$



The Cayley graph of S_3



and a tree in it

The object groups of the free groupoid $F(T)$ on the graph T are all trivial. For each $g \in S_3$ let $h_0 g$ denote the unique element of $F(T)(g,1)$, so that the section $\sigma : S_3 \to F$ of φ is given by $h_0 g = (g,(\sigma g)^{-1}), g \in G$. Then for each $(g,u) : g \to g'$ in $F(\widetilde{X})$, we set $\rho(g,u) = (h_0(g))^{-1}(g,u)h_0(g')$. Let $\widetilde{N} = F(\widetilde{X})(1)$, which is mapped isomorphically by p_2 to $N = \delta_2(C(R)) = \mathrm{Ker}\ \varphi$. Thus the tree T determines a retraction $\rho : F(\widetilde{X}) \to \widetilde{N}$.

Let D be the set of edges of $\widetilde{X}$ which do not lie in T. Then the set $\rho(D)$ is a set of free generators of the group $\widetilde{N}$, and $p_1\rho(D)$ is a set of free generators of the group N. Let $\eta_i = \rho\theta_i, i = 1, \ldots, 7$.

In order to define $h_1 : F(\widetilde{X}) \to C(\widetilde{R})(1)$ we need only to give its values on the generators (see 9.1). We give these by $h_1(\tau) = 1$ if $\tau \in T$ and for $\theta \in D$, we let $h_1(\theta)$ be an element of $C(\widetilde{R})(1)$ which is mapped by $\widetilde{\delta}_2$ to $\rho(\theta)$. Then h_1 satisfies (2), and also $h_1(\theta) = h_1(\rho(\theta))$.

In our example of S_3, we define h_1 on $\rho(D)$, and so on $F(\widetilde{X})$, as follows:

$$\begin{array}{lll}
\eta_1 = \rho(\varphi y, y) & h_1\eta_1 = (1,s) & = (1,s) \\
\eta_2 = \rho(\varphi x, y) & h_1\eta_2 = (1,t)(1,s)^{-1} & = (1,ts^{-1}) \\
\eta_3 = \rho(\varphi xy, y) & h_1\eta_3 = (1,s)(1,t)^{-1}(\varphi x, s)^{(1,x)^{-1}} & = (1, st^{-1}s^{x^{-1}}) \\
\eta_4 = \rho(\varphi yx, y) & h_1\eta_4 = (\varphi y, t)^{(1,y)^{-1}} & = (1, t^{y^{-1}}) \\
\eta_5 = \rho(\varphi x^2, y) & h_1\eta_5 = (\varphi x^2, s)^{(\varphi x^2, x)}((\varphi x^2, t)^{-1})^{(\varphi x^2, x)} & = (1, s^x(t^{-1})^x) \\
\eta_6 = \rho(\varphi yx, x) & h_1\eta_6 = (\varphi y, r)^{(1,y)^{-1}} & = (1, r^{y^{-1}}) \\
\eta_7 = \rho(\varphi x, x) & h_1\eta_7 = (1,r) & = (1,r)
\end{array}$$

This ensures that $\widetilde{\delta}_2 h_1(\eta_i) = \eta_i, i = 1, 2, \ldots, 7$.

In order to calculate the identities among relations, we now need to express $\rho\widetilde{\delta}_2\alpha$ in terms of the η_i for all $\alpha \in \widetilde{R}$. Then according to the previous section we can obtain an identity among relations for each $\alpha \in \widetilde{R}$, namely

$$p_2\left((h_1\widetilde{\delta}_2\alpha)^{-1}\alpha^{h_0\beta\alpha}\right).$$

The results of these calculations are given in the table which follows. The order of writing the identities is chosen so that the first four give our eventual minimal set of generators, the next six give trivial identities, and the last has the most difficult verification of its dependence on the first four.

Table 1

generator		$\rho\widetilde{\delta}_2\alpha_i$	$\gamma_i = p_2\left(h_1(\widetilde{\delta}_2\alpha_i)^{-1}\alpha_i^{h_0\beta\alpha_i}\right)$	
α_1	$(\varphi x^2, r)$	η_7	γ_1	$r^{-1}r^x$
α_2	$(\varphi y, s)$	η_1	γ_2	$s^{-1}s^{y^{-1}}$
α_3	$(\varphi x^2, s)$	$\eta_5\eta_4$	γ_3	$(t^{-1})^{y^{-1}}t^x$
α_4	$(\varphi x, t)$	$\eta_7\eta_5\eta_6\eta_3$	γ_4	$(s^{-1})^{x^{-1}}ts^{-1}(r^{-1})^{y^{-1}}t^x(s^{-1})^x r^{-1}t^{x^{-1}}$
α_5	$(1, r)$	η_7	γ_5	1
α_6	$(1, s)$	η_1	γ_6	1
α_7	$(1, t)$	$\eta_2\eta_1$	γ_7	1
α_8	$(\varphi x, s)$	$\eta_2\eta_3$	γ_8	1
α_9	$(\varphi y, t)$	η_5	γ_9	1
α_{10}	$(\varphi y, r)$	η_6	γ_{10}	1
α_{11}	$(\varphi x, r)$	η_7	γ_{11}	$r^{-1}r^{x^{-1}}$
α_{12}	$(\varphi xy, r)$	η_6	γ_{12}	$(r^{-1})^{y^{-1}}r^{xy^{-1}}$
α_{13}	$(\varphi yx, r)$	η_6	γ_{13}	$(r^{-1})^{y^{-1}}r^{x^{-1}y^{-1}}$
α_{14}	$(\varphi xy, s)$	$\eta_3\eta_2$	γ_{14}	$(s^{-1})^{yxy^{-1}}s^{xy^{-1}}$
α_{15}	$(\varphi xy, t)$	$\eta_1\eta_2$	γ_{15}	$(t^{-1})^{y^{-2}}t^{xy^{-1}}$
α_{16}	$(\varphi x^2, t)$	η_5	γ_{16}	$(t^{-1})^{y^{-1}}t^x$
α_{17}	$(\varphi yx, s)$	$\eta_4\eta_5$	γ_{17}	$t^x(s^{-1})^x(t^{-1})^{y^{-1}}s^{x^{-1}y^{-1}}$
α_{18}	$(\varphi yx, t)$	$\eta_6\eta_3\eta_7\eta_5$	γ_{18}	$t^x(s^{-1})^x r^{-1}(s^{-1})^{x^{-1}}ts^{-1}(r^{-1})^{y^{-1}}t^{x^{-1}y^{-1}}$

We now let I consist of elements $\bar{\alpha}_i$ in one-to-one correspondence with the α_i, and let $C_3(I)$ be the free G-module on I. Define $\delta_3\bar{\alpha}_i$ to be the value $\gamma_i \in C_2$ given in the fourth column of the table. Let $\widetilde{I} = G \times I$, and let $C_3(\widetilde{I})$ be the free $\widetilde{G}$-module on $\widetilde{I}$. We define

$$h_2(\alpha_i) = (1, \bar{\alpha}_i), i = 1, \dots, 18.$$

Then we have

$$h_1(\widetilde{\delta}_2\alpha_i)^{-1}\alpha_i^{h_0\beta(\alpha_i)} = (1, \gamma_i) = \widetilde{\delta}_3 h_2\alpha_i, i = 1, \dots, 18.$$

So we have extended our covering complex and its contracting homotopy by one stage.

However, we can in fact omit all of the α_i except the first four, because of the trivial identities $\gamma_5, \dots, \gamma_{10}$, and the further relations given in Table 2 below. We give the verification for the last two further relations, the others being trivial or easy.

We note that

$$\begin{aligned}
\gamma_{17} &= t^x(s^{-1})^x(t^{-1})^{y^{-1}}s^{x^{-1}y^{-1}} \\
&= t^x(t^{-1})^{y^{-1}}(s^{-1})^{xyy^{-1}x^{-1}y^{-1}x^{-1}y^{-1}}s^{x^{-1}y^{-1}} && \text{by the crossed module rules} \\
&= (t^{-1})^{y^{-1}}t^{y^{-1}x^{-1}y^{-1}}((s^{-1})^{y^{-1}}s)^{x^{-1}y^{-1}} \\
&= (t^{-1})^{y^{-1}}t^x((s^{-1})^{y^{-1}}s)^{x^{-1}y^{-1}} && \text{since } t^{(\delta_2 t^{-1})x} = t^x \\
&= \gamma_3(\gamma_2^{-1})^{x^{-1}y^{-1}}
\end{aligned}$$

In order to verify the further identity for γ_{18}, we consider the abelianisation $C(R)^{\mathrm{ab}}$, which is isomorphic to the free S_3-module on R. The difference $\gamma_{18} - \gamma_4$ in $C(R)^{\mathrm{ab}}$ is

$$t.(\varphi(x^{-1}y^{-1}) - \varphi(x^{-1})) = t.(\varphi(xy) - 1)\varphi(x^2).$$

Since the module of identities is mapped injectively into $C(R)^{\mathrm{ab}}$, [16], and in $C(R)^{\mathrm{ab}}$ we have $\gamma_3 = t.(\varphi(x) - \varphi(y^{-1}))$, the result follows. So we have a set of four generators for the module of identities for this presentation of S_3, of which the first three given belong to the root module (see [16] for an account of this).

Let J consist of the elements $\bar{\alpha}_i, i = 1, \dots, 4$, and let $C_3(J)$ be the free G-module on J, with the restriction to it of the boundary δ_3. Let $\mathsf{r} : C_3(I) \to C_3(J)$ be be the G-module morphism defined by $\mathsf{r}(\bar{\alpha}_i) = \bar{\alpha}_i, i = 1, \dots, 4,\ \mathsf{r}(\bar{\alpha}_i) = 0, i = 5, \dots, 10$ and otherwise as in Table 2 below, so that $\delta_3\mathsf{r} = \delta_3$. Note that $C_3(I)$ is an S_3-module, and so we write it additively as a group, and use . for the group action. To simplify the notation we write these acting elements as words in the generators x, y.

Table 2

γ_i	identity	$\mathsf{r}\bar{\alpha}_i$
γ_{11}	$=(\gamma_1^{-1})^{x^{-1}}$	$-\bar{\alpha}_1.x^2$
γ_{12}	$=\gamma_1^{y^{-1}}$	$\bar{\alpha}_1.y$
γ_{13}	$=(\gamma_1^{-1})^{x^{-1}y^{-1}}$	$-\bar{\alpha}_1.yx$
γ_{14}	$=\gamma_2^{yxy^{-1}}$	$\bar{\alpha}_2.x^2$
γ_{15}	$=\gamma_3^{y^{-1}}$	$\bar{\alpha}_3.y$
γ_{16}	$=\gamma_3$	$\bar{\alpha}_3$
γ_{17}	$=\gamma_3(\gamma_2^{-1})^{x^{-1}y^{-1}}$	$\bar{\alpha}_3-\bar{\alpha}_2.yx$
γ_{18}	$=\gamma_4\gamma_3^{yx^2}$	$\bar{\alpha}_4+\bar{\alpha}_3.xy$

Define $\mathsf{r}' : C_3(\widetilde{I}) \to C_3(\widetilde{J}), (g,d) \mapsto (g, \mathsf{r}d)$, and define $h_2' = \mathsf{r}'h_2 : C(\widetilde{R}) \to C_3(\widetilde{J})$. Then we have for $i = 1, 2, \dots, 18$

$$\widetilde{\delta}_3 h_2' \alpha_i = h_1(\widetilde{\delta}_2\alpha_i)^{-1}\alpha_i^{h_0\beta\alpha_i}$$

and so we have a contracting homotopy up to this level.

Note that we now have 24 generators $\tilde{d} = (g, \bar{\alpha}_i), g \in S_3, i = 1, \dots, 4$ of $C_3(\widetilde{J})$ and we can proceed to the next stage, to obtain identities between identities corresponding to each of these generators, namely

$$p_3(-h_2\widetilde{\delta}_3\tilde{d} + \tilde{d}^{h_0\beta\tilde{d}})$$

for $\tilde{d} = (g, \bar{\alpha}_i), g \in S_3, i = 1, \dots, 4$. This requires another table. In order to show how the calculations go, we next carry out one intermediate calculation, and one full calculation. Further details of the calculations required for the table are omitted, but are available on request.

Recall that h_2' is a groupoid morphism, and kills the action of $F(\widetilde{X})$. So, for example,

$$\begin{aligned} h_2'(\varphi y, t^x) &= h_2'((\varphi y x^{-1}, t)^{(\varphi xy, x)}) \\ &= h_2'(\varphi xy, t) & &= h_2'(\alpha_{15}) \\ &= (1, \bar{\alpha}_3^{y^{-1}}) & &= (1, \bar{\alpha}_3^{\varphi y^{-1}}) \\ &= (1, \bar{\alpha}_3^{y}). \end{aligned}$$

So we have

$$\begin{aligned} &- h_2'\widetilde{\delta}_3(\varphi yx, \bar{\alpha}_4) + (\varphi yx, \bar{\alpha}_4)^{(yx, x^{-1}y^{-1})} \\ &= -h_2'(\varphi yx, (s^{-1})^{x^{-1}}ts^{-1}(r^{-1})^{y^{-1}}t^x(s^{-1})^x r^{-1}t^{x^{-1}}) + (1, \bar{\alpha}_4^{x^{-1}y^{-1}}) \\ &= -(1, -\bar{\alpha}_2.x^2 + \bar{\alpha}_4 + \bar{\alpha}_3.xy - \bar{\alpha}_3 + \bar{\alpha}_2.yx - \bar{\alpha}_1 - \bar{\alpha}_2 + \bar{\alpha}_1.yx + \bar{\alpha}_3.y - \bar{\alpha}_4.yx) \\ &= (1, \bar{\alpha}_1.(1-yx) + \bar{\alpha}_2.(1+x^2-yx) + \bar{\alpha}_3.(1-y-xy) + \bar{\alpha}_4.(-1+yx)) \end{aligned}$$

Some of the identities in Table 3 might seem as surprising to others as they were to the authors. There is a process for checking that these are identities among identities as follows.

We are required to check that δ_3 of some combination u of the $\bar{\alpha}_i$ is zero. Certainly each $\delta_3\bar{\alpha}_i$ is an identity among relations, and hence so is the corresponding linear combination u. Therefore u is 0 if and only if it maps to 0 in the abelianised group $C(R)^{\mathrm{ab}}$, which is freely generated as a $\mathbb{Z}S_3$ module by

the elements r, s, t. Thus we determine the coefficients of these elements for the image of u in $C(R)^{\mathrm{ab}}$, and it is straightforward to check that these are zero. This is analogous to a previous calculation.

Table 3

generator		identity $= p_3\left(-h_2\widetilde{\delta}_3\theta_i + \theta_i^{h_0\beta\theta_i}\right)$	
ξ_1	$(1, \bar{\alpha}_1)$	μ_1	0
ξ_2	$(1, \bar{\alpha}_2)$	μ_2	0
ξ_3	$(1, \bar{\alpha}_3)$	μ_3	0
ξ_4	$(1, \bar{\alpha}_4)$	μ_4	0
ξ_5	$(\varphi x, \bar{\alpha}_1)$	μ_5	0
ξ_6	$(\varphi x, \bar{\alpha}_2)$	μ_6	0
ξ_7	$(\varphi x, \bar{\alpha}_3)$	μ_7	$\bar{\alpha}_3.(y + x^2)$
ξ_8	$(\varphi x, \bar{\alpha}_4)$	μ_8	$\bar{\alpha}_1.(y - x^2) + \bar{\alpha}_4.(x^2 - 1)$
ξ_9	$(\varphi x^2, \bar{\alpha}_1)$	μ_9	$\bar{\alpha}_1.(1 + x + x^2)$
ξ_{10}	$(\varphi x^2, \bar{\alpha}_2)$	μ_{10}	$\bar{\alpha}_2.(1 + y)x$
ξ_{11}	$(\varphi x^2, \bar{\alpha}_3)$	μ_{11}	$\bar{\alpha}_3.(1 + yx)x$
ξ_{12}	$(\varphi x^2, \bar{\alpha}_4)$	μ_{12}	$\bar{\alpha}_1.(1 - yx) + \bar{\alpha}_4.(x - 1)$
ξ_{13}	$(\varphi y, \bar{\alpha}_1)$	μ_{13}	0
ξ_{14}	$(\varphi y, \bar{\alpha}_2)$	μ_{14}	$\bar{\alpha}_2.(1 + y)$
ξ_{15}	$(\varphi y, \bar{\alpha}_3)$	μ_{15}	0
ξ_{16}	$(\varphi y, \bar{\alpha}_4)$	μ_{16}	$\bar{\alpha}_2.(1 + x^2 - yx) + \bar{\alpha}_3.(1 - y - xy) - \bar{\alpha}_4.(1 - y)$
ξ_{17}	$(\varphi yx, \bar{\alpha}_1)$	μ_{17}	0
ξ_{18}	$(\varphi yx, \bar{\alpha}_2)$	μ_{18}	0
ξ_{19}	$(\varphi yx, \bar{\alpha}_3)$	μ_{19}	$\bar{\alpha}_3.(1 + yx)$
ξ_{20}	$(\varphi yx, \bar{\alpha}_4)$	μ_{20}	$\bar{\alpha}_1.(1 - yx) + \bar{\alpha}_2.(1 + x^2 - yx) + \bar{\alpha}_3.(1 - y - xy) + \bar{\alpha}_4.(-1 + yx)$
ξ_{21}	$(\varphi yx^2, \bar{\alpha}_1)$	μ_{21}	$\bar{\alpha}_1.(1 + x + x^2)y$
ξ_{22}	$(\varphi yx^2, \bar{\alpha}_2)$	μ_{22}	$\bar{\alpha}_2.(1 + y)x^2$
ξ_{23}	$(\varphi yx^2, \bar{\alpha}_3)$	μ_{23}	0
ξ_{24}	$(\varphi yx^2, \bar{\alpha}_4)$	μ_{24}	$\bar{\alpha}_1.(y - x^2) + \bar{\alpha}_2.(1 + x^2 - yx) + \bar{\alpha}_3.(1 + x^2 - xy) + \bar{\alpha}_4.(-1 + xy)$

We next reduce this to a smaller, and clearly minimal, set of identities among identities, as in the following table.

Table 4

generator	definition/further identity
μ_9	$\bar{\alpha}_1.(1+x+x^2)$
μ_{14}	$\bar{\alpha}_2.(1+y)$
μ_{19}	$\bar{\alpha}_3.(1+yx)$
μ_{12}	$\bar{\alpha}_1.(1-yx)+\bar{\alpha}_4.(x-1)$
μ_{16}	$\bar{\alpha}_2.(1+x^2-yx)+\bar{\alpha}_3.(1-y-xy)-\bar{\alpha}_4.(1-y)$
μ_{21}	$=\mu_9.y$
μ_{10}	$=\mu_{14}.x$
μ_{22}	$=\mu_{14}.x^2$
μ_{11}	$=\mu_{19}.x$
μ_7	$=\mu_{19}.x^2$
μ_8	$=\mu_9.(-1+y)+\mu_{12}.(x+1)$
μ_{20}	$=\mu_9.(1-y)+\mu_{12}.(x+1)y+\mu_{16}$
μ_{24}	$=\mu_{12}.y+\mu_{16}+\mu_{19}.x^2$

Let K be set with elements $\bar{\mu}_9, \bar{\mu}_{14}, \bar{\mu}_{19}, \bar{\mu}_{12}, \bar{\mu}_{16}$, let $C_4(K)$ be the free G-module on K, and let $\delta_4 : C_4(K) \to C_3(J)$ be given by the first four line of the second column of Table 4. Then the sequence $C_4(K) \to C_3(J) \to C(R)$ is exact and we have extended our crossed resolution by one further step. Hence we have a presentation of the G-module $\pi(\mathcal{P})$.

Such a crossed resolution has been extended by two further steps, but with different choices, in [23].

As explained in the Introduction, this example is chosen as one which illustrates the method, which has non trivial calculations but also is perhaps the largest example of this type which one would care to do by hand. The major problems are the calculation of h_1, i.e. representing a set of group generators of $N = \mathrm{Ker}\ \varphi$ as consequences of the relators, and more seriously, calculating minimal generating subsets of sets of generators of submodules of free $\mathbb{Z}G$-modules, as well as finding the relations giving all the generators in terms of the smaller set. The first problem is dealt with in [25] and the second in [24].

3 Presentations of groupoids

The category of groupoids will be written Gpd. Our convention for groupoids is that the composite of arrows $a : x \to y, b : y \to z$ is written $ab : x \to z$.

The theory of groupoids may be thought of as an algebraic analogue of the theory of groups, but based on directed graphs rather than on sets. For some discussion of the philosophy of this, see [5].

3.1 Free groupoids

The term *graph* will always mean what is usually called a directed graph. A *graph* X consists of two sets $Arr(X), Ob(X)$, of arrows and objects respectively of X, and two functions $s, t : Arr(X) \to Ob(X)$, called the *source* and *target* maps. A *morphism* $f : X \to Y$ of graphs consists of two functions

$Arr(X) \to Arr(Y), Ob(X) \to Ob(Y)$, which commute with the source and target maps. This defines the category Graph.

A basic construction in any algebraic theory is that of free objects. For groups, the free group functor $F : \mathsf{Set} \to \mathsf{Group}$ is left adjoint to the forgetful functor $\mathsf{Group} \to \mathsf{Set}$. In the case of groupoids, we may define the *free groupoid* functor to be the left adjoint $F : \mathsf{Graph} \to \mathsf{Gpd}$ to the forgetful functor $U : \mathsf{Gpd} \to \mathsf{Graph}$ giving the underlying graph UG of a groupoid G, namely forgetting the composition, the identity function $Ob(G) \to G$, and the inverse map $G \to G$. So if X is a graph, then the free groupoid $F(X)$ on X consists of a graph morphism $i : X \to UF(X)$ which is universal for morphisms from X to the underlying graph of a groupoid.

The set of objects of $F(X)$ may be identified with $Ob(X)$. There are several ways of explicitly constructing the set of arrows of $F(X)$. The usual way is as equivalence classes of *composable* words

$$w = (x_1, \varepsilon_1) \dots (x_n, \varepsilon_n), n \geqslant 0, x_i \in Arr(X), \varepsilon = \pm$$

together with empty words $(\,)_a, a \in Ob(X)$, where the word w is composable means that $t(x_i, \varepsilon_i) = s(x_{i+1}, \varepsilon_{i+1}), i = 1 \dots n-1$, where

$$s(x,\varepsilon) = \begin{cases} sx & \text{if } \varepsilon = +, \\ tx & \text{if } \varepsilon = -, \end{cases} \qquad t(x,\varepsilon) = \begin{cases} tx & \text{if } \varepsilon = +, \\ sx & \text{if } \varepsilon = -. \end{cases}$$

The equivalence relation on words, and the composition, to obtain the free groupoid is defined in a manner analogous to the usual definition of free group, and the graph morphism $i : X \to F(X)$ sends an arrow x to $[x]$, the equivalence class of the word $(x, +)$.

A groupoid G is called *connected* if $G(a,b)$ is non empty for all $a, b \in Ob(G)$. The maximal connected subgroupoids of G are called the *(connected) components* of G.

If a is an object of the groupoid G, then the set $G(a,a)$ inherits a group structure from the composition on G, and this is called the *object group* of G at a and is written also $G(a)$. The groupoid G is called *simply connected* if all its object groups are trivial. If it is connected and simply connected, it is called 1-*connected*, or a *tree groupoid*.

A standard example of a tree groupoid is the *indiscrete*, or *square*, groupoid $I(S)$ on a set S. This has object set S and arrow set $S \times S$, with $s, t : S \times S \to S$ being the first and second projections. The composition on $I(S)$ is given by

$$(a,b)(b,c) = (a,c), a, b, c \in S.$$

A graph X is called *connected* if the free groupoid $F(X)$ on X is connected, and is called a *forest* if every object group $F(X)(a)$ of $F(X), a \in Ob(X)$, is trivial. A connected forest is called a *tree*. If X is a tree, then $F(X)$ is a tree groupoid.

3.2 Retractions

Let G be a connected groupoid. Let a_0 be an object of G. For each object a of G choose an arrow $\tau a : a \to a_0$, with $\tau a_0 = 1_{a_0}$. Then an isomorphism

$$\varphi : G \to G(a_0) \times I(Ob(G))$$

is given by $g \mapsto ((\tau a)^{-1} g(\tau b), (a, b)), g \in G(a, b), a, b \in Ob(G)$. The composition of φ with the projection yields a morphism $\rho : G \to G(a_0)$ which we call a *deformation retraction*, since it is the identity on $G(a_0)$ and is in fact homotopic to the identity morphism of G, though we do not elaborate on this fact here.

It is also standard [4, 8.1.5] that a connected groupoid G is isomorphic to the free product groupoid $G(a_0) * T$ where $a_0 \in Ob(G)$ and T is any wide, tree subgroupoid of G. The importance of this is as follows.

Suppose that X is a graph which generates the connected groupoid G. Then X is connected. Choose a maximal tree T in X. Then T determines for each a_0 in $Ob(G)$ a retraction $\rho_T : G \to G(a_0)$ and the isomorphisms

$$G \cong G(a_0) * I(Ob(G)) \cong G(a_0) * F(T)$$

show that a morphism $G \to K$ from G to a groupoid K is completely determined by a morphism of groupoids $G(a_0) \to K$ and a graph morphism $T \to K$ which agree on the object a_0.

We shall use later the following proposition, which is a special case of [4, 6.7.3]:

Proposition 3.1 *Let G, H be groupoids with the same set of objects, and let $\varphi : G \to H$ be a morphism of groupoids which is the identity on objects. Suppose that G is connected and $a_0 \in Ob(G)$. Choose a retraction $\rho : G \to G(a_0)$. Then there is a retraction $\sigma : H \to H(a_0)$ such that the following diagram, where φ' is the restriction of φ:*

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\varphi"'] & G(a_0) \arrow[d, "\varphi'"] \\ H \arrow[r, "\sigma"'] & H(a_0) \end{tikzcd} \tag{8}$$

is commutative and is a pushout of groupoids.

3.3 Normal subgroupoids and quotient groupoids

Let G be a groupoid. A subgroupoid N of G is called *normal* if N is wide in G (i.e. $Ob(N) = Ob(G)$) and for any objects a, b of G and g in $G(b, a)$, $g^{-1}N(b)g = N(a)$.

Let $\varphi : G \to H$ be a morphism of groupoids. Then Ker φ is the wide subgroupoid of G whose elements are all g in G such that φg is an identity of H is a normal subgroupoid of G. If $Ob(f)$ is injective then Ker φ is totally disconnected, i.e. (Ker φ)$(a, b) = \emptyset$ if $a \neq b$.

A morphism $\varphi : G \to H$ is said to *annihilate* a subgraph X of G if $\varphi(X)$ is a discrete subgroupoid of H. Thus Ker φ is the largest subgroupoid of G annihilated by φ. The next proposition gives the existence of quotient groupoids.

Proposition 3.2 *Let N be a totally disconnected, normal subgroupoid of G. Then there is a groupoid G/N and a morphism $p : G \to G/N$ such that p annihilates N and is universal for morphisms from G which annihilate N.*

Proof We define $Ob(G/N) = Ob(G)$. If $a, b \in Ob(G)$ we define $(G/N)(a, b)$ to consist of all cosets $N(a)g, g \in G(a, b)$. The multiplication of G is inherited by G/N, which becomes a groupoid.

The morphism $p : G \to G/N$ is the identity on objects, and on elements is defined by $g \mapsto N(sg)g$. Clearly p is a morphism and Ker $p = N$.

The remainder of the proof is clear. $\Box$

We call G/N a *quotient groupoid* of G.

3.4 Presentations of groupoids

We now consider relations in a groupoid. Suppose given for each object a of the groupoid G a set $R(a)$ of elements of $G(a)$— thus R can be regarded as a wide, totally disconnected subgraph of G. The *normal closure* $N(R)$ of R is the smallest wide normal subgroupoid of G which contains R. This obviously exists since the intersection of any family of normal subgroupoids of G is again a normal subgroupoid of G. Further, $N(R)$ is totally disconnected since the family of object groups of any normal subgroupoid N of G is again a normal subgroupoid of G.

Alternatively, $N = N(R)$ can be constructed explicitly. Let a be an object of G. By a *consequence* of R at a is meant either the identity of G at a, or any product

$$\tau = g_1^{-1} r_1^{\varepsilon_1} g_1 \dots g_n^{-1} r_n^{\varepsilon_n} g_n, \tag{9}$$

in which $n \geqslant 1$, $g_i \in G(a_i, a)$ for some object a_i of G, $\varepsilon_i = \pm 1$ and r_i is an element of $R(a_i)$. Clearly, the set $N(a)$ of consequences of R at a is a subgroup of $G(a)$ and the family $N = (N(a) : a \in Ob(G))$ of these groups is a totally disconnected normal subgroupoid of G containing R. Clearly $N = N(R)$.

The projection $p : G \to G/N(R)$ has the following universal property: *if $f : G \to H$ is any morphism which annihilates R then there is a unique morphism $f : G/N(R) \to H$ such that $fp = f$.* We call $G/N(R)$ *the groupoid G with the relations $r = 1, r \in R$.*

In applications, we are often given G, R as above and wish to describe the object groups of $G/N(R)$. These are determined by the following result.

Proposition 3.3 [4, 8.3.3] *Let G be connected, let $a_0 \in Ob(G)$ and let $\rho : G \to G(a_0)$ be a deformation retraction. Let $H = G/N(R)$. Then $H(a_0)$ is isomorphic to the group $G(a_0)$ with the relations*

$$\rho(r) = 1, r \in R.$$

Proof The proof follows from Proposition 3.1, with $H = G/N$ and $\varphi = p : G \to G/N$ the quotient morphism. Details are given in [4]. $\Box$

4 Crossed modules and free crossed modules over groupoids

The theory of crossed modules and free crossed modules is due to Whitehead [37]. Expositions are given in for example [16, 28]. In order to obtain an algebraic model of universal covers, we need the corresponding definitions for the groupoid case, due to Brown and Higgins in [10].

Let Φ be a groupoid. A *crossed Φ-module* consists of:

(i) a totally disconnected groupoid M with the same object set as Φ;

(ii) a morphism $\mu : M \to \Phi$ of groupoids which is the identity on objects; and

(iii) an action of the groupoid Φ on the right of the groupoid M *via* μ.

This last condition means that if $x \in \Phi(a,b), m \in M(p)$, then $m^x \in M(b)$ and the usual laws of an action apply, namely $m^1 = m, (m^x)^y = m^{xy}, (mn)^x = m^x n^x$ whenever the terms are defined.

The axioms for a crossed module are:

CM1) $\mu(m^x) = x^{-1}(\mu m)x$,

CM2) $n^{-1}mn = m^{\mu n}$,

for all $m, n \in M, x \in \Phi$ and whenever the terms are defined.

Such a crossed Φ-module is written (M, μ, Φ) or $\mu : M \to \Phi$, or simply as M.

A *morphism* from a crossed module $\mu : M \to \Phi$ to a crossed module $\nu : N \to \Psi$ consists of a pair of morphisms of groupoids $f : \Phi \to \Psi,\ g : M \to N$ such that $\nu g = f\mu$ and $g(m^x) = (gm)^{fx}$ whenever m^x is defined. This yields the category XMod of crossed modules and their morphisms.

There is also a category PXMod of precrossed modules, in which the axiom CM2) is dropped. The inclusion of categories $\mathsf{XMod} \to \mathsf{PXMod}$ has a left adjoint constructed as follows.

Let $\mu : M \to \Phi$ be a precrossed module. By a *Peiffer element*, or *twisted commutator*, is meant an element

$$\langle m, n \rangle = m^{-1}n^{-1}mn^{\mu m}$$

where $m, n \in M(p)$ for some object p. As in the group case (see [16, Proposition 2, p.158]) one proves that the Peiffer elements generate a normal Φ-invariant subgroupoid $\langle M, M \rangle$ of M, and the quotient groupoid, $M^{ass} = M/\langle M, M \rangle$, with the induced morphism $\mu' : M^{ass} \to \Phi$, inherits the structure of crossed module. This *associated crossed module* gives the reflection from the category PXMod of precrossed modules to the category XMod of crossed modules as required.

Let Φ be a groupoid, let R be a totally disconnected graph with the same object set as Φ, and let $w : R \to \Phi$ be a graph morphism which is the identity on objects. We define the *free crossed module on* w to be a crossed module $\partial : C(w) \to \Phi$ together with a graph morphism $\bar{w} : R \to C(w)$ such that:

(i) $\partial \bar{w} = w$;

(ii) if $\mu : M \to \Phi$ is a crossed module and $g : R \to M$ is a graph morphism over the identity on objects such that $\mu g = w$, then there is a unique morphism $g' : C(R) \to M$ of crossed Φ-modules such that $g'\bar{w} = g$.

Free crossed modules over groups were defined and constructed by Whitehead [37], and an exposition is given in [16]. The analogous construction for groupoids is as follows.

Let $w : R \to \Phi$ be given as above. One first forms the free groupoid $H(w)$ on the totally disconnected graph Y with object set $Ob(\Phi)$ where $Y(p)$ consists of pairs (r, u) such that $r \in R(q), u \in \Phi(q, p)$. Let $\partial' : H(w) \to \Phi$ be given by $(r, u) \mapsto u^{-1}(wr)u$, and let Φ operate on $H(w)$ by $(r, u)^v = (r, uv)$. This yields the *free precrossed module* on w, and the free crossed module is the associated crossed module $\partial : C(w) = H(w)^{ass} \to \Phi$.

Notice that the image $\partial(C(w))$ is the normal closure $N(wR)$ of wR in Φ.

It is useful to see this construction as a special case of the *induced crossed modules* of [9] (but for the groupoid case), which can be regarded as arising from a 'change of base' [6]. That is, $C(w)$ is isomorphic to the crossed module $\omega_* F(R)$ induced from the identity crossed module $1 : F(R) \to F(R)$ by the morphism $\omega : F(R) \to \Phi$ determined by $w : R \to \Phi$. Further, we have a pushout of crossed modules

$$\begin{tikzcd}
(1,0,F(R)) \arrow[r, "{(1,\omega)}"] \arrow[d] & (1,0,\Phi) \arrow[d] \\
(F(R),1,F(R)) \arrow[r] & (C(w),\partial,\Phi)
\end{tikzcd}$$

This allows a link with the 2-dimensional Van Kampen Theorem of [9] (or rather, with the groupoid version formulated in all dimensions in [11]), to obtain a proof of a groupoid version of a well known theorem of Whitehead [37], as follows:

Theorem 4.1 *Let U_0 be a subset of the space U and suppose the space V is obtained from U by attaching 2-cells by maps of pairs $f_r : (S^1, 1) \to (U, U_0), r \in R$. Then the family of second relative homotopy groups $\pi_2(V, U, p), p \in U_0$ form the free crossed module over the fundamental groupoid $\pi_1(U, U_0)$ on the graph morphism $w : R \to \pi_1(U, U_0)$ given by $wr = (f_r)_*(\iota)$, where ι here denotes a generator of the fundamental group $\pi_1(S^1, 1)$.*

5 Crossed complexes

The basic geometric example of a crossed complex is the *fundamental crossed complex* πX_* of a filtered space

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X.$$

Here $\pi_1 X_*$ is the fundamental groupoid $\pi_1(X_1, X_0)$ and for $n \geqslant 2$, $\pi_n X_*$ is the family of relative homotopy groups $\pi_n(X_n, X_{n-1}, p)$ for all $p \in X_0$. These come equipped with the standard operations of $\pi_1 X_*$ on $\pi_n X_*$ and boundary maps $\delta : \pi_n X_* \to \pi_{n-1} X_*$. The axioms for crossed complexes are those universally satisfied for this example.

The definition of a crossed complex generalises to the case of a set of base points definitions given by Blakers [2] (under the term 'group system') and Whitehead [37], under the term 'homotopy system' (except that he restricted also to the free case). We recall this general definition from [10].

A *crossed complex* C (over a groupoid) is a sequence of morphisms of groupoids over C_0

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\delta_n"] \arrow[d, "\beta"] & C_{n-1} \arrow[r] \arrow[d, "\beta"] & \cdots \arrow[r] & C_2 \arrow[r, "\delta_2"] \arrow[d] & C_1 \arrow[d, shift right, "\delta^0"'] \arrow[d, shift left, "\delta^1"] \\
& C_0 & C_0 & & C_0 & C_0 .
\end{tikzcd}$$

Here $\{C_n\}_{n \geqslant 2}$ is a family of groups with base point map β, and δ^0, δ^1 are the source and targets for the groupoid C_1. We further require given an operation of the groupoid C_1 on each family of groups C_n for $n \geqslant 2$ such that:

(i) each δ_n is a morphism over the identity on C_0;

(ii) $C_2 \to C_1$ is a crossed module over C_1;

(iii) C_n is a C_1-module for $n \geqslant 3$;

(iv) $\delta : C_n \to C_{n-1}$ is an operator morphism for $n \geqslant 3$;

(v) $\delta\delta : C_n \to C_{n-2}$ is trivial for $n \geqslant 3$;

(vi) δC_2 acts trivially on C_n for $n \geqslant 3$.

Because of axiom (iii) we shall write the composition in C_n additively for $n \geqslant 3$, but we will use multiplicative notation in dimensions 1 and 2.

Let C be a crossed complex. Its *fundamental groupoid* $\pi_1 C$ is the quotient of the groupoid C_1 by the normal, totally disconnected subgroupoid δC_2. The rules for a crossed complex give C_n, for $n \geqslant 3$, the induced structure of $\pi_1 C$-module.

A *morphism* $f : C \to D$ of crossed complexes is a family of groupoid morphisms $f_n : C_n \to D_n$ $(n \geqslant 0)$ which preserves all the structure. This defines the category Crs of crossed complexes. The fundamental groupoid now gives a functor $\pi_1 : \mathsf{Crs} \to \mathsf{Gpd}$. This functor is left adjoint to the functor $i : \mathsf{Gpd} \to \mathsf{Crs}$ where for a groupoid G the crossed complex iG agrees with G in dimensions 0 and 1, and is otherwise trivial.

An *m-truncated* crossed complex C consists of all the structure defined above but only for $n \leqslant m$. In particular, an m-truncated crossed complex is for $m = 0, 1, 2$ simply a set, a groupoid, and a crossed module respectively.

6 Covering morphisms of groupoids and crossed complexes

For the convenience of readers, and to fix the notation, we recall here the basic facts on covering morphisms of groupoids.

Let G be a groupoid. For each object a of G the *star* of a in G, denoted by $\mathrm{St}_G\, a$, is the union of the sets $G(a, b)$ for all objects b of G, i.e. $\mathrm{St}_G a = \{g \in G : sg = a\}$. A morphism $p : \widetilde{G} \to G$ of groupoids is a *covering morphism* if for each object $\widetilde{a}$ of $\widetilde{G}$ the restriction of p

$$\mathrm{St}_{\widetilde{G}}\, \widetilde{a} \to \mathrm{St}_G\, p\widetilde{a}$$

is bijective. In this case $\widetilde{G}$ is called a *covering groupoid of G*.

A basic result for covering groupoids is *unique path lifting*. That is, let $p : \widetilde{G} \to G$ be a covering morphism of groupoids, and let $(g_1, g_2, \ldots, g_n)$ be a sequence of composable elements of G. Let $\tilde{a} \in Ob(\widetilde{G})$ be such that $p\tilde{a}$ is the starting point of g_1. Then there is a unique composable sequence $(\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_n)$ of elements of $\widetilde{G}$ such that $\tilde{g}_1$ starts at $\tilde{a}$ and $p\tilde{g}_i = g_i, i = 1, \ldots, n$.

If G is a groupoid, the category GpdCov/G of coverings of G has as objects the covering morphisms $p : H \to G$ and has as arrows (morphisms) the commutative diagrams of morphisms of groupoids,

where p and q are covering morphisms,

$$\begin{tikzcd} H \arrow[rr, "f"] \arrow[dr, "p"'] & & K \arrow[dl, "q"] \\ & G & \end{tikzcd}$$

By a result of [4], f also is a covering morphism. It is convenient to write such a diagram as a triple (f, p, q). The composition in GpdCov/G is then given as usual by

$$(g, q, r)(f, p, q) = (gf, p, r).$$

It is a standard result (see for example [27, 3]) that the category GpdCov/G is equivalent to the functor category Set^G. This is useful for constructing covering morphisms of the groupoid G. For example, if a is an object of the transitive groupoid G, then the groupoid G operates on the family of stars $\mathrm{St}_G\, a$, and the associated covering morphism $\widetilde{G} \to G$ defines the *universal cover* $\widetilde{G}$ of the groupoid G. In particular, this gives the universal covering groupoid of a group.

We now give the generalisation of this notion to crossed complexes.

Definition 6.1 [29] A morphism $p : \widetilde{C} \to C$ of crossed complexes is a *covering morphism* if

(i) the morphism $p_1 : (\widetilde{C}_1, \widetilde{C}_0) \to (C_1, C_0)$ is a covering morphism of groupoids;

(ii) for each $n \geqslant 2$ and $\widetilde{x} \in \widetilde{C}_0$, the morphism of groups $p_n : \widetilde{C}_n(\widetilde{x}) \to C_n(p\widetilde{x})$ is an isomorphism.

In such case we call $\widetilde{C}$ a *covering crossed complex* of C.

This definition may also be expressed in terms of the unique covering homotopy property. For more details (but there with emphasis on fibrations) see [15].

Proposition 6.2 *Let $p : \widetilde{C} \to C$ be a covering morphism of crossed complexes and let $\tilde{a} \in Ob(\widetilde{C})$. Let $a = p\tilde{a}$, and let $K = p_0^{-1}(a) \subseteq Ob(\widetilde{C})$. Then p induces isomorphisms $\pi_n(\widetilde{C}, \tilde{a}) \to \pi_n(C, a)$ for $n \geqslant 2$ and a sequence*

$$1 \to \pi_1(\widetilde{C}, \tilde{a}) \to \pi_1(C, a) \to K \to \pi_0(\widetilde{C}) \to \pi_0(C)$$

which is exact in the sense of the exact sequence of a fibration of groupoids.

The comment about exactness has to do with operations on the pointed sets: see [3, 4]. The proof of the proposition is easy and is omitted.

The following result gives a basic geometric example of a covering morphism of crossed complexes.

Theorem 6.3 *Let X_* and Y_* be filtered spaces and let $f : X \to Y$ be a covering map of spaces such that for each $n \geqslant 0$, $f_n : X_n \to Y_n$ is also a covering map with $X_n = f^{-1}(Y_n)$. Then $\pi f : \pi X_* \to \pi Y_*$ is a covering morphism of crossed complexes.*

Proof By a result of [4], $\pi f : \pi_1 X_1 \to \pi_1 Y_1$ is a covering morphism of groupoids. Since $X_0 = f^{-1}(Y_0)$, the restriction of $\pi_1 f$ to $\pi_1(X_1, X_0) \to \pi_1(Y_1, Y_0)$ is also a covering morphism of groupoids. Now for each $n \geqslant 2$ and for each $x_0 \in X_0$, $f_* : \pi_n(X_n, X_{n-1}, x_0) \to \pi_n(Y_n, Y_{n-1}, p(x_0))$ is an isomorphism (see for example, [30]). $\square$

Here is an important method of constructing new covering morphisms.

Proposition 6.4 *Let $p : \widetilde{C} \to C$ be a covering morphism of crossed complexes. Then the induced morphism $\pi_1(p) : \pi_1 \widetilde{C} \to \pi_1 C$ is a covering morphism of groupoids.*

Proof Let $\tilde{x} \in \widetilde{C}_0$. We will show that $p'_{\tilde{x}} : \mathrm{St}_{\pi_1 \widetilde{C}}\, \tilde{x} \to \mathrm{St}_{\pi_1 C}\, p\tilde{x}$ is bijective. Let $[a] \in \mathrm{St}_{\pi_1 C}\, p\tilde{x}$, where $a \in \mathrm{St}_C p\tilde{x}$. Since p is a covering morphism, there exists a unique $\tilde{a}$ of $\mathrm{St}_{\widetilde{C}} \tilde{x}$ such that $p\tilde{a} = a$. So $p'_{\tilde{x}}[\tilde{a}] = [a]$ and thus $p'_{\tilde{x}}$ is surjective.

Now suppose that $p'_{\tilde{x}}[\tilde{a}] = p'_{\tilde{x}}[\tilde{b}]$. Then $(p\tilde{b})^{-1} p\tilde{a} \in \delta C_2(p\tilde{x})$ which implies that $(p\tilde{b})^{-1}(p\tilde{a}) = \delta p\tilde{c}$ for a unique $\tilde{c} \in \widetilde{C}_2(\tilde{x})$. Because p is a covering morphism, we need only show that $(\tilde{b})^{-1}\tilde{a} = \delta\tilde{c}$. This follows by star injectivity. Therefore $p'_{\tilde{x}}$ is injective and so is bijective. Hence $\pi_1(p)$ is a covering morphism of groupoids. $\square$

Let C be a crossed complex. We write CrsCov/C for the full subcategory of the slice category Crs/C whose objects are the covering morphisms of C.

Proposition 6.5 *Suppose given a pullback diagram of crossed complexes*

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\bar{f}"] \arrow[d, "\bar{q}"'] & \widetilde{E} \arrow[d, "q"] \\ C \arrow[r, "f"'] & E \end{tikzcd}$$

in which q is a covering morphism. Then $\bar{q}$ is a covering morphism.

We omit the proof. The groupoid case is done in [4, 9.7.6]. See also [8] for uses of pullbacks of covering morphisms of groupoids.

Our next result is the analogue for covering morphisms of crossed complexes of a classical result for covering maps of spaces [4, 9.6.1].

Theorem 6.6 *If C is a crossed complex, then the functor $\pi_1 : \mathsf{Crs} \to \mathsf{Gpd}$ induces an equivalence of categories*

$$\pi'_1 : \mathsf{CrsCov}/C \to \mathsf{GpdCov}/(\pi_1 C).$$

Proof If $p : \widetilde{C} \to C$ is a covering morphism of crossed complexes, then $\pi_1 p : \pi_1 \widetilde{C} \to \pi_1 C$ is a covering morphism of groupoids, by Proposition 6.4. Since π_1 is a functor, we also obtain the functor π'_1. To prove π'_1 is an equivalence of categories, we construct a functor $\rho : \mathsf{GpdCov}/(\pi_1 C) \to \mathsf{CrsCov}/C$ and prove that there are equivalences of functors $1 \simeq \rho\pi'_1$ and $1 \simeq \pi'_1\rho$.

Let C be a crossed complex, and let $q : D \to \pi_1 C$ be a covering morphism of groupoids. Let $\widetilde{C}$ be given by the pullback diagram in the category of crossed complexes:

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\bar{\varphi}"] \arrow[d, "\bar{q}"'] & iD \arrow[d, "q"] \\ C \arrow[r, "\varphi"'] & i\pi_1 C \end{tikzcd} \tag{10}$$

By proposition 6.5, $\bar{q} : \widetilde{C} \to C$ is a covering morphism of crossed complexes.

We define the functor ρ by $\rho(q) = \bar{q}$, and extend ρ in the obvious way to morphisms.

The natural transformation $\pi_1'\rho \simeq 1$ is defined on a covering morphism $q : D \to \pi_1 C$ to be the composite morphism

$$\lambda : \pi_1(\widetilde{C}) \xrightarrow{\pi_1(\bar{\varphi})} \pi_1(iD) \cong D$$

where $\bar{\varphi} : \widetilde{C} \to iD$ is given in diagram (10). The proof that λ is an isomorphism is simple and is left to the reader.

To prove that $1 \simeq \rho\pi_1'$, we show that the following diagram is a pullback:

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\widetilde{\varphi}"] \arrow[d, "q"'] & i\pi_1\widetilde{C} \arrow[d, "q' = i\pi_1(q)"] \\ C \arrow[r, "\varphi"'] & i\pi_1 C \end{tikzcd}$$

This is clear in dimension 0 and in dimensions $\geqslant 2$. For the case of dimension 1, let $c : x \to y$ in C, and $[\tilde{c}] \in (\pi_1\widetilde{C})(\tilde{x}, \tilde{y})$ be such that $q[\tilde{c}] = \varphi(c)$. Then there exists a unique $\tilde{c}' : \tilde{x} \to \tilde{y}$ such that $\widetilde{\varphi}(\tilde{c}') = [\tilde{c}]$ and $\bar{q}(\tilde{c}') = c$. Now, $\bar{q}(\tilde{c}\delta\widetilde{C}_2(\tilde{x})) = \varphi(c) = c\delta C_2(x)$. This implies that $(\bar{q}\tilde{c})\delta C_2(x) = c\delta C_2(x)$. So $\bar{q}(\tilde{c}) = c(\delta c_2)$ for some $c_2 \in C_2(x)$. Therefore there exists a unique $\tilde{c}_2 \in \widetilde{C}_2(\tilde{x})$ covering c_2, and $\bar{q}(\tilde{c}(\delta\tilde{c}_2)^{-1}) = c$. So the above diagram is a pullback and thus we have proved that $1 \simeq \rho\pi_1'$. This proves the equivalence of the two categories. $\square$

7 Covering morphisms and colimits

In this section we give a result due to Howie [29, Theorem 5.1] which we use to prove covering crossed complexes of free crossed complexes are free.

Theorem 7.1 *Let $p : A \to B$ be a morphism of crossed complexes. Then p is a fibration if and only if the pullback functor $p^* : \mathsf{Crs}/B \to \mathsf{Crs}/A$ has a right adjoint.*

As a consequence we get the following.

Corollary 7.2 *If $p : A \to B$ is a covering morphism of crossed complexes, then $p^* : \mathsf{Crs}/B \to \mathsf{Crs}/A$ preserves all colimits.*

8 Coverings of free crossed complexes

We recall here a definition from [12]. A *free basis* for a crossed complex C consists of subgraphs X_n of C_n for all $n \geqslant 1$ such that C_1 is the free groupoid on X_1, C_2 is the free crossed C_1-module on the restriction $\delta_2' : X_2 \to C_1$, and for $n \geqslant 3$, C_n is the free $\pi_1 C$-module on X_n.

Following [15] we write $\mathbb{C}(n)$ for the crossed complex freely generated by one generator c_n in dimension n. So $\mathbb{C}(0)$ is the singleton set $\{1\}$; $\mathbb{C}(1)$ is the groupoid $\mathcal{I}$ which has two objects 0, 1 and non-identity elements $c_1 : 0 \to 1$ and $c_1^{-1} : 1 \to 0$; and for $n \geqslant 2, \mathbb{C}(n)$ is in dimensions n and $n-1$ an infinite cyclic group with generators c_n and δc_n respectively, and is otherwise trivial. Thus if C is a crossed complex, then an element $c \in C_n$ is completely specified by a morphism $\hat{c} : \mathbb{C}(n) \to C$ such that $\hat{c}(c_n) = c$, and $\delta(c) = \hat{c}(\delta c_n)$.

Let $\mathbb{S}(n-1)$ be the subcomplex of $\mathbb{C}(n)$ which agrees with $\mathbb{C}(n)$ up to dimension $n-1$ and is trivial otherwise. If $\mathbf{E}^n$ and $\mathbf{S}^{n-1}$ denote the skeletal filtrations of the standard n-ball and $(n-1)$-sphere, where $E^0 = \{0\}, S^{-1} = \emptyset, E^1 = I = \{0,1\} \cup e^1, S^0 = \{0,1\}$, and for $n \geqslant 2, E^n = \{1\} \cup e^{n-1} \cup e^n, S^{n-1} = \{1\} \cup e^{n-1}$, then it is clear that for all $n \geqslant 0, \mathbb{C}(n) \cong \pi \mathbf{E}^n$ and $\mathbb{S}(n-1) \cong \pi \mathbf{S}^{n-1}$.

We now model for crossed complexes the process for spaces known as attaching cells. Let A be any crossed complex. A sequence of morphisms $j_n : C^{n-1} \to C^n$ may be defined with $C^0 = A$ by choosing any family of morphisms $\mathbb{S}(m_\lambda - 1) \to C^{n-1}$ for any $\lambda \in \Lambda_n$ and any m_λ, and forming the pushout

$$\begin{tikzcd} \coprod_{\lambda \in \Lambda_n} \mathbb{S}(m_\lambda - 1) \arrow[r] \arrow[d] & C^{n-1} \arrow[d] \\ \coprod_{\lambda \in \Lambda_n} \mathbb{C}(m_\lambda) \arrow[r] & C^n . \end{tikzcd} \tag{11}$$

Let $C = \mathrm{colim}_n C^n$, and let $j : A \to C$ be the canonical morphism. The morphism $j : A \to C$ is called a *relatively free crossed complex morphism.* If A is empty, then we call C a *free* crossed complex.

The importance of the definition is as follows:

8.1 *If C is a free crossed complex on X_*, then a morphism $f : C \to D$ can be constructed inductively provided one is given the values $f_n x \in D_n, x \in X_n, n \geqslant 0$ provided the following geometric conditions are satisfied: (i) $\delta^\alpha f_1 x = f_0 \delta^\alpha x, x \in X_1, \alpha = 0, 1$; (ii) $\beta f_n(x) = f_0(\beta x), x \in X_n, n \geqslant 2$; (iii) $\delta_n f_n(x) = f_{n-1}\delta_n(x), x \in X_n, n \geqslant 2$.*

Notice that in (iii), f_{n-1} has to be defined on all of C_{n-1} before this condition can be verified.

We now show that freeness can be lifted to covering crossed complexes.

Theorem 8.2 *Suppose given a pullback square of crossed complexes*

$$\begin{tikzcd} \widetilde{A} \arrow[r, "\bar{j}"] \arrow[d, "p'"'] & \widetilde{C} \arrow[d, "p"] \\ A \arrow[r, "j"'] & C \end{tikzcd}$$

in which p is a covering morphism and $j : A \to C$ is relatively free. Then $\bar{j} : \widetilde{A} \to \widetilde{C}$ is relatively free.

Proof We suppose given the sequence of diagrams 11. Let $\hat{C}^n = p^{-1}(C^n)$. By corollary 7.2, the following diagram is a pushout:

$$\begin{tikzcd}
p^*\left(\coprod_{\lambda\in\Lambda_n} \mathbb{S}(m_\lambda - 1)\right) \arrow[r] \arrow[d] & \hat{C}^{n-1} \arrow[d] \\
p^*\left(\coprod_{\lambda\in\Lambda_n} \mathbb{C}(m_\lambda)\right) \arrow[r] & \hat{C}^n.
\end{tikzcd}$$

Since p is a covering morphism, we can write $p^*\left(\coprod_{\lambda\in\Lambda_n} \mathbb{C}(m_\lambda)\right)$ as $\coprod_{\lambda\in\widetilde{\Lambda}_n} \mathbb{C}(m_\lambda)$ for a suitable $\widetilde{\Lambda}_n$. This completes the proof. □

Corollary 8.3 *Let $p : \widetilde{C} \to C$ be a covering morphism of crossed complexes. If C is free on X_*, then $\widetilde{C}$ is free on $p^{-1}(X_*)$.*

A similar result to Corollary 8.3 applies in the m-truncated case.

The significance of these results is as follows. We start with an m-truncated free crossed resolution C of a group G, so that we are given $\varphi : C_1 \to G$, and C is free on X_*, where X_n is defined only for $n \leqslant m$. Our extension process of section 9 will start by constructing the universal cover $p : \widetilde{C} \to C$ of C; this is the covering crossed complex corresponding to the universal covering groupoid $p_0 : \widetilde{G} \to G$. By the results above, $\widetilde{C}$ is the free crossed complex on $p^{-1}(X_*)$. It also follows from Proposition 6.2 that the induced morphism $\widetilde{\varphi} : \widetilde{C} \to \widetilde{G}$ makes $\widetilde{C}$ a free crossed resolution of the contractible groupoid $\widetilde{G}$. Hence $\widetilde{C}$ is an acyclic and hence, since it is free, also a contractible crossed complex.

We now see the general context for the diagram (1) of section 1 and the exposition there.

9 Homotopies

We follow the conventions for homotopies in [13]. Thus a homotopy $f^0 \simeq f$ of morphisms $f^0, f : C \to D$ of crossed complexes is a pair (h, f) where h is a family of functions $h_n : C_n \to D_{n+1}$ with the following properties, in which βc for $c \in C$ is c, if $c \in C_0$, is $\delta^1 c$, if $c \in C_1$, and is x if $c \in C_n(x)$, $n \geqslant 2$. So we require [13, (3.1)]:

$$\begin{aligned}
\beta h_n(c) &= \beta f(c) && \text{for all } c \in C; && (12)\\
h_1(cc') &= h_1(c)^{fc'}\ h_1(c') && \text{if } c, c' \in C_1 \text{ and } cc' \text{ is defined;} && (13)\\
h_2(cc') &= h_2(c) + h_2(c') && \text{if } c, c' \in C_2 \text{ and } cc' \text{ is defined;} && (14)\\
h_n(c + c') &= h_n(c) + h_n(c') && \text{if } c, c' \in C_n,\ n \geqslant 3 \text{ and } c + c' \text{ is defined;} && (15)\\
h_n(c^{c_1}) &= (h_n c)^{fc_1} && \text{if } c \in C_n, n \geqslant 2,\ c_1 \in C_1, \text{ and } c^{c_1} \text{ is defined.} && (16)
\end{aligned}$$

Then f^0, f are related by [13, (3.14)]

$$f^0(c) = \begin{cases} \delta^0 h_0 c & \text{if } \ c \in C_0, \\ (h_0\delta^0 c)(fc)(\delta_2 h_1 c)(h_0 \delta^1 c)^{-1} & \text{if } \ c \in C_1, \\ \{(fc)(h_1\delta_2 c)(\delta_3 h_2 c)\}^{(h_0\beta c)^{-1}} & \text{if } \ c \in C_2, \\ \{fc + h_{n-1}\delta_n c + \delta_{n+1} h_n c\}^{(h_0\beta c)^{-1}} & \text{if } \ c \in C_n,\ n \geqslant 3. \end{cases} \qquad (17)$$

The following is important for our computations. We saw in 8.1 that a morphism is specified by its values on a graded set of free generators. We now show that the same is true for homotopies.

9.1 *If C is a free crossed complex on a generating family $X_n, n \geqslant 0$, then a homotopy $(h, f) : f^0 \simeq f : C \to D$ is specified by the values $fx \in D_n, hx \in D_{n+1}, x \in X_n, n \geqslant 0$ provided only that the following geometric conditions hold:*

$$\begin{aligned} &\delta^0 fx = f\delta^0 x, \delta^1 fx = f\delta^1 x, x \in X_1, \delta fx = f\delta x, x \in X_n, n \geqslant 2, \\ &\beta fx = f\beta x, x \in X_n, n \geqslant 1, \beta hx = \beta fx, x \in X_n, n \geqslant 0. \end{aligned} \tag{18}$$

Proof All but the last condition are those given for the construction of f in 8.1. The final fact we need is that for $n \geqslant 2$ the f_1-morphism h_n is defined by its values on the generators in X_n, and this is standard. □

This result is another aspect of the facts that a homotopy $C \to D$ can also be regarded as a morphism $\mathbb{C}(1) \otimes C \to D$, where the tensor product is defined in [13], and the tensor product of free complexes is free is proved in [15].

From this we can deduce formulae for a retraction. Suppose then in the above formulae we take $C = D,\ f^0 = 1_C,\ f = 0$ where 0 denotes the constant morphism on C mapping everything to a base point 0. Then the homotopy $h : 1 \simeq 0$ must satisfy

$$\beta h_n c = 0 \qquad \text{if } c \in C, \tag{19}$$

$$\delta^0 h_0 c = c \qquad \text{if } c \in C_0, \tag{20}$$

$$\delta_2 h_1 c = (h_0\delta^0 c)^{-1} c\, (h_0 \delta^1 c) \qquad \text{if } c \in C_1, \tag{21}$$

$$\delta_3 h_2 c = (h_1\delta_2 c)^{-1} c^{h_0\beta c} \qquad \text{if } c \in C_2, \tag{22}$$

$$\delta_{n+1} h_n c = -h_{n-1}\delta_n c + c^{h_0\beta c} \qquad \text{if } c \in C_n, n \geqslant 3, \tag{23}$$

$$h_n(c^{c_1}) = (h_n c) \qquad \text{if } c \in C_n, n \geqslant 2,\ c_1 \in C_1,\ \text{and } c^{c_1} \text{ is defined.} \tag{24}$$

Further, in this case h_1 is a morphism by (13) and for $n \geqslant 2$, h_n is by (15) a morphism which by (16) trivialises the operations of C_1. All these conditions are necessary and sufficient for h to be a contracting homotopy.

An *m-truncated crossed complex* C is a crossed complex as earlier except that C_n and δ_n are defined only for $n \leqslant m$. Similarly, for a contracting homotopy h of an m-truncated crossed complex C, we have h_n defined only for $n < m$ and the above conditions hold where they make sense.

Our main result is now rather formal and straightforward to prove. It is to extend the pair (C, h) of a partial free crossed resolution C and partial contracting homotopy h of the universal cover of C by one step. Hence the process can be continued indefinitely.

Theorem 9.2 *Let $m \geqslant 1$ and let C be an m-truncated free crossed resolution of a group G. Let $p : \widetilde{C} \to C$ be the universal cover of C so that $\widetilde{C}$ is an m-truncated free crossed resolution of the universal covering groupoid $p_0 : \widetilde{G} \to G$ of G. Let h be a partial contracting homotopy of $\widetilde{C}$. Suppose also that $\widetilde{C}_m$ is free on $\widetilde{X}_m$.*

Let $\widetilde{X}_m \to X_{m+1}, \tilde{x} \mapsto x$, be a bijection to a set X_{m+1} disjoint from $\widetilde{X}_m$. Define an extension $e(C)$ of C to an $(m+1)$-truncated free crossed complex as follows:
For $m = 1$, let $C_2 = e(C)_2$ be the free crossed C_1-module on X_2 with $\delta_2 : X_2 \to C_1$ given by

$$\delta_2 x = p_1 \left((h_0 \widetilde{\delta}^0 \tilde{x})^{-1} \, \tilde{x} \, (h_0 \widetilde{\delta}^1 \tilde{x}) \right), \, \tilde{x} \in \widetilde{X}_1. \tag{25}$$

For $m = 2$ let $C_3 = e(C)_3$ be the free G-module on X_3 with $\delta_3 : e(C)_3 \to C_2$ defined by

$$\delta_3 x = p_2 \left((h_1 \widetilde{\delta}_2 \tilde{x})^{-1} \, \tilde{x}^{h_0 \beta \tilde{x}} \right), \qquad \tilde{x} \in \widetilde{X}_2. \tag{26}$$

For $m \geqslant 3$ let $C_{m+1} = e(C)_{m+1}$ be the free G-module on X_{m+1} with $\delta_{m+1} : C_{m+1} \to C_m$ defined by

$$\delta_{m+1} x = p_m \left(-h_{m-1} \widetilde{\delta}_m \tilde{x} + \tilde{x}^{h_0 \beta \tilde{x}} \right), \, \tilde{x} \in \widetilde{X}_m. \tag{27}$$

Let $e(p) : e(\widetilde{C}) \to e(C)$ be the induced covering morphism, extending p by $p_{m+1} : \widetilde{C}_{m+1} \to C_{m+1}$.

Define $h_m : \widetilde{C}_m \to \widetilde{C}_{m+1}$ on the basis $\widetilde{X}_m$ by $h_m(\tilde{x}) = (1, x)$. Then this extension $e(h)$ of h is a contracting homotopy of $e(\widetilde{C})$. Hence $e(C)$ is an $(m+1)$-truncated free crossed resolution of G.

If further there is a subset Y of X_{m+1} such that $\delta_{m+1} Y$ also generates $\ker \delta_m$, and a retraction $\xi : C_{m+1} \to C_{m+1}(Y)$ is given such that $\delta_{m+1} \xi(x) = \delta_{m+1}(x)$ for all $x \in X_{m+1}$, and ξ is a G-morphism for $m \geqslant 2$, and a crossed C_1-morphism for $m = 1$, then we may replace C_{m+1} by $C_{m+1}(Y)$ and h_m by ξh_m to again get an extension of the pair (C, h) by one step.

Proof The fact that we have a contracting homotopy is immediate from the definitions. It follows that $e(\widetilde{C})$ is exact, and so $e(C)$ is aspherical with $\pi_1(e(C)) = G$. □

Corollary 9.3 *Under the assumptions of Theorem 9.2, if $m = 1$ then Ker $\varphi : C_1 \to G$ is generated as a normal subgroup of C_1 by the elements:*

$$p_1 \left((h_0 \widetilde{\delta}^0 \tilde{x})^{-1} \, \tilde{x} \, (h_0 \widetilde{\delta}^1 \tilde{x}) \right), \, \tilde{x} \in \widetilde{X}_1. \tag{28}$$

For $m \geqslant 2$, Ker $(\delta_m : C_m \to C_{m-1})$ is generated as a G-module by the elements:

$$p_2 \left((h_1 \widetilde{\delta}_2 \tilde{x})^{-1} \, \tilde{x}^{h_0 \beta \tilde{x}} \right), \qquad \tilde{x} \in \widetilde{X}_2, \quad \text{if } m = 2, \tag{29}$$

$$p_m \left(-h_{m-1} \widetilde{\delta}_m \tilde{x} + \tilde{x}^{h_0 \beta \tilde{x}} \right), \, \tilde{x} \in \widetilde{X}_m, \quad \text{if } m \geqslant 3. \tag{30}$$

We have now finally justified the process set out in section 1 and illustrated with an example in section 2.

In papers to follow we will give implementations of these methods, and so a wider range of calculations.

10 Examples

10.1 The standard crossed resolution of a group

The standard crossed resolution of a group was defined by Huebschmann in [31] and applied also in for example [17, 35]. Here we show how this resolution arises from our procedure.

We start with a group G and let $C_1 = F(G)$, the free group on the set G, with generators written $[a],\ a \in G$. Let $\varphi : C_1 \to G$ be the canonical morphism. This has a section $\sigma : G \to F(G)$, $a \mapsto [a], a \neq 1, 1 \mapsto 1$. This defines $h_0 : G \to \widetilde{C}_1$, $a \mapsto (a, [a]^{-1})$.

The Cayley graph of this presentation has arrows $(a, [b]) : a \to ab$ so that $h_0(a)^{-1}(a, [b])h_0(ab) = (1, [a][b][ab]^{-1})$. So we may take C_2 to be the free crossed C_1-module on elements $[a, b]$ and define $\delta_2 : C_2 \to C_1$ by

$$\delta_2[a, b] = [a][b][ab]^{-1}.$$

Then in the universal cover we can define $h_1 : \widetilde{C}_1 \to \widetilde{C}_2(1)$ by $(a, [b]) \mapsto (1, [a, b])$.

Theorem 10.1 *There is a free crossed $C_*(G)$ resolution of a group G in which $C_n(G)$ is free on the set G^n with generators written $[a_1, a_2, \dots, a_n], a_i \in G$, with contracting homotopy of the universal cover given by $(a, [a_1, a_2, \dots, a_n]) \mapsto (1, [a, a_1, a_2, \dots, a_n])$, and boundary $\delta_n : C_n(G) \to C_{n-1}(G)$ given by δ_2 as above,*

$$\delta_3[a, b, c] = [a, bc][ab, c]^{-1}[a, b]^{-1}[b, c]^{[a]^{-1}},$$

and for $n \geqslant 4$

$$\begin{aligned}\delta_n[a_1, a_2, \dots, a_n] = [a_2, \dots, a_n]^{a_1^{-1}} + \sum_{i=1}^{n-1}(-1)^i[a_1, a_2, \dots, a_{i-1}, a_ia_{i+1}, a_{i+2}, \dots, a_n] +& \\ + (-1)^n[a_1, a_2, \dots, a_{n-1}].& \qquad (31)\end{aligned}$$

Proof We first verify

$$\begin{aligned}\widetilde{\delta}_3h_2(a, [b, c]) &= h_1\widetilde{\delta}_2(a, [b, c])^{-1}(a, [b, c])^{(a,[a]^{-1})} \\ &= h_1(1, a, [b][c][bc]^{-1})^{-1}(1, [b, c]^{[a]^{-1}}) \\ &= h_1((a, [b])(ab, [c])(abc, [bc]^{-1}))^{-1}(1, [b, c]^{[a]^{-1}}) \\ &= h_1((a, [b])(ab, [c])(a, [bc])^{-1})^{-1}(1, [b, c]^{[a]^{-1}}) \\ &= (1, [a, bc][ab, c]^{-1}[a, b]^{-1}[b, c]^{[a]^{-1}}).\end{aligned}$$

In order to have a contracting homotopy we require for $n \geqslant 3$

$$\begin{aligned}&\widetilde{\delta}_{n+1}h_n(a_1, [a_2, \dots, a_{n+1}]) \\ &= -h_{n-1}\widetilde{\delta}_n(a_1, [a_2, \dots, a_{n+1}]) + (1, [a_2, \dots, a_{n+1}]^{a_1^{-1}}) \\ &= (1, [a_2, \dots, a_{n+1}]^{a_1^{-1}} + \sum_{i=1}^{n}(-1)^i[a_1, \dots, a_ia_{i+1}, \dots, a_{n+1}] + (-1)^{n+1}[a_1, a_2, \dots, a_n]).\end{aligned}$$

This completes the proof that the family h_n give a contracting homotopy and so that $C_*(G)$ is a resolution.

□

10.2 A small crossed resolution of finite cyclic groups

This is the resolution given in [18]. Here we shall describe its universal cover and a contracting homotopy. We would like to thank A. Heyworth for discussions on this section.

We write C for the (multiplicative) infinite cyclic group with generator x, and C_r for the finite cyclic group of order r with generator t. Let $\varphi : \mathsf{C} \to \mathsf{C}_r$ be the morphism sending x to t. We show how the inductive procedure given earlier recovers the small free crossed resolution of C_r together with a contracting homotopy of the universal cover.

Let $p_0 : \widetilde{\mathsf{C}}_r \to \mathsf{C}_r$ be the universal covering morphism, and let $p_1 : \widetilde{\mathsf{C}} \to \mathsf{C}$ be the induced cover of C. Then $\widetilde{\mathsf{C}}$ is the free groupoid on the Cayley graph $\widetilde{X}$ pictured as follows:

$$\begin{tikzcd}
1 \arrow[r, "{(1,x)}"'] & t \arrow[r, "{(t,x)}"'] & t^2 \arrow[r, "{(t^2,x)}"'] & \cdots\cdots \arrow[r] & t^{n-2} \arrow[r, "{(t^{n-2},x)}"'] & t^{n-1} \arrow[lllll, bend right, "{(t^{n-1},x)}"']
\end{tikzcd}$$

A section $\sigma : \mathsf{C}_r \to \mathsf{C}$ of φ is given by $t^i \mapsto x^i, i = 0, \ldots , r-1$, and this defines

$$h_0 : \mathsf{C}_r \to \widetilde{F}_1,\ t^i \mapsto (t^i, x^{-i}).$$

It follows that for $i = 0, \ldots , r-1$

$$h_0(t^i)^{-1}(t^i, x)h_0(t^{i+1}) = \begin{cases} (1, 1) & \text{if } i \neq r-1, \\ (1, x^r) & \text{if } i = r-1. \end{cases}$$

So we take a new generator x_2 for F_2 with $\delta_2 x_2 = x_r$ and set

$$h_1(t^i, x) = \begin{cases} (1, 1) & \text{if } i \neq r-1, \\ (1, x_2) & \text{if } i = r-1. \end{cases}$$

Then for all $i = 0, \ldots , r-1$ we have

$$\widetilde{\delta}_2 h_1(t^i, x) = h_0(t^i)^{-1}(t^i, x)h_0(t^{i+1}),$$

and it follows that

$$\begin{aligned} h_1(t^i, x^r) &= h_1((t^i, x)(t^{i+1}, x) \ldots (t^{i+r-1}, x)) \\ &= (1, x_2). \end{aligned} \tag{32}$$

Hence

$$\begin{aligned} -h_1\widetilde{\delta}_2(t^i, x_2) + (t^i, x_2).x^{-i} &= (1, -x_2) + (1, x_2.t^{-i}) \\ &= (1, x_2.(t^{r-i} - 1)) \end{aligned}$$

This gives us 0 for $i = 0$, and $(1, x_2.(t-1))$ for $i = r-1$. Let $N(i) = 1 + t + \cdots + t^{i-1}$, so that $t^{r-i} - 1 = (t-1)N(r-i)$ for $i = 1, \ldots, r-1$. Hence we can take a new generator x_3 for F_3 with $\delta_3 x_3 = x_2.(t-1)$ and define

$$h_2(t^i, x_2) = \begin{cases} (1,0) & \text{if } i = 0, \\ (1, x_3.N(r-i)) & \text{if } 0 < i \leqslant r-1. \end{cases}$$

Now we find that if we evaluate

$$-h_2\widetilde{\delta}_2(t^i, x_3) + (1, x_3.t^{-i}) = -h_2((t^{i-1}, x_2).t + (t^i, x_2)) + (1, x_3.t^{-i})$$

we obtain for $i = 0$

$$-h_2(t^{r-1}, x_2) + (1, x_3) = (1, 0),$$

for $i = 1$

$$0 + h_2(t, x_2) + (1, x_3.t^{r-1}) = (1, x_3.(N(r-1) + t^{r-1})) = (1, x_3.N(r))$$

and otherwise

$$(1, x_3(-N(r-i+1) + N(r-i) + t^{r-i})) = (1, 0).$$

Thus we take a new generator x_4 for F_4 with $\delta_4 x_4 = x_3.N(r)$ and

$$h_3(t^i, x_3) = \begin{cases} (1, x_4) & \text{if } i = 1, \\ (1, 0) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} -h_3\widetilde{\delta}_4(t^i, x_4) + (1, x_4.t^{-i}) &= -h_3(t^i, x_3.N(r)) + (1, x_4.t^{-i}) \\ &= -h_3(1, x_3.N(r).t^{-i}) + (1, x_4.t^{-i}) \\ &= (1, x_4.(t^{r-i} - 1)). \end{aligned}$$

Thus we are now in a periodic situation and we have the theorem:

Theorem 10.2 *A free crossed resolution F_* of C_r may be taken to have single free generators x_n in dimension $n \geqslant 1$ with $\varphi(x_1) = t$, and*

$$\delta_n(x_n) = \begin{cases} x_1^r & \text{if } n = 2, \\ x_{n-1}.(t-1) & \text{if } n > 1, n \text{ odd}, \\ x_{n-1}.N(r) & \text{if } n > 2, n \text{ even}. \end{cases}$$

A contracting homotopy on $\widetilde{F}_*$ is given by $h_0(t^i) = (t^i, x_1^{-i})$ and for $n > 1$

$$h_n(t^i, x_n) = \begin{cases} (1,0) & \text{if } n = 1, i \neq r-1, \\ (1, x_2) & \text{if } n = 1, i = r-1, \\ (1,0) & \text{if } n \text{ even}, n \geqslant 2, i = 0, \\ (1, x_{n+1}.N(r-i)) & \text{if } n \text{ even}, n \geqslant 2, 0 < i \leqslant r-1, \\ (1,0) & \text{if } n \text{ odd}, n > 1, i \neq 1, \\ (1, x_{n+1}) & \text{if } n \text{ odd}, n > 1, i = 1. \end{cases}$$

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